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Another Look at Climate Sensitivity

Ilya Zaliapin[1] **and Michael Ghil**[2]

[1]Department of Mathematics and Statistics, University of Nevada, Reno, USA. E-mail: zal@unr.edu.
[2]Geosciences Department and Laboratoire de Météorologie Dynamique (CNRS and IPSL), Ecole Normale Supérieure, Paris, FRANCE, and Department of Atmospheric & Oceanic Sciences and Institute of Geophysics & Planetary Physics, University of California, Los Angeles, USA. E-mail: ghil@atmos.ucla.edu.

Abstract. We analyze here a classical energy balance model (EBM) with nonlinear feedbacks and revisit in this context a recent claim that — in a model-independent sense — the climate system may exhibit an asymmetric, large-amplitude response to normally distributed forcing; such a response would imply irreducible uncertainty in climate change predictions. We show that equilibrium climate sensitivity in all generality does not support such an intrinsic indeterminacy.

EBMs, though, do exhibit a saddle-node bifurcation that gives rise to several steady-state climates; this behavior is, furthermore, supported by results from more realistic, nonequilibrium climate models. In a truly nonlinear setting, indeterminacy in the size of the response is associated with the vicinity to such a bifurcation. We discuss this bifurcation in the EBM context and recall some well-known results about the stability and equilibrium sensitivity of such models. It is shown here, moreover, that small disturbances cannot result in a large-amplitude response, unless the system is at or near the bifurcation point.

Our EBM suggests that the current Earth climate is relatively close to this bifurcation, and a moderate decrease of the global radiative input may bring the Earth to a “snowball” state; results of general circulation models point in the same direction. We discuss briefly how the distance to the bifurcation may be related to the Earth’s ice-albedo feedback.

Keywords: Climate sensitivity, energy balance models, global warming, stability analysis, bifurcations

Correspondence to: Ilya Zaliapin (zal@unr.edu)

1 Introduction and motivation

1.1 Climate sensitivity and its implications

Systems with feedbacks are an efficient mathematical tool for modeling a wide range of natural phenomena; Earth’s climate is one of the most prominent examples. Stability and sensitivity of feedback models is, accordingly, a traditional topic of theoretical climate studies [e.g., Cess (1976); Ghil (1976); Crafoord and Källén (1978); Schlesinger (1985, 1986); Cess *et al.* (1989)]. Roe and Baker (2007) have recently advocated existence of intrinsically large sensitivities in a nonlinear equilibrium model with multiple feedbacks. Specifically, they argued that a small, normally distributed feedback may lead to large-magnitude, asymmetrically distributed values of the system’s response.

Such a property, if valid, would have serious implications for climate dynamics (Allen and Frame, 2007) and for modeling of complex systems in general (Watkins and Freeman, 2008). In this paper, we revisit the dynamical behavior of a general, equilibrium climate model with nonlinear feedbacks, and focus subsequently on a simple energy-balance model (EBM). We notice that the main, technical part of Roe and Baker’s argument is well-known in the climate literature [e.g., Schlesinger (1985, 1986)] and thus it seems useful to review the associated assumptions and possible interpretations of this result.

We rederive below the key equation of Roe and Baker (2007) and proceed in Section 2 with a more self-consistent version of sensitivity analysis for a nonlinear model. This analysis is applied in Section 3 to a zero-dimensional EBM.

Concluding remarks follow in Section 4.

1.2 The Roe and Baker analysis

We follow here Roe and Baker (2007) and assume the following general setup. Let the net radiation R at the top of the atmosphere be related to the corresponding average temperature T at the Earth's surface by $R = R(T)$. Furthermore, we assume that there exists a feedback $\alpha = \alpha(T)$, which is affected by the temperature change and which can affect the radiative balance. Hence, one can write $R = R\,(T, \alpha(T))$.

To study how a small change $\Delta\,R$ in the radiation is related to the corresponding temperature change $\Delta\,T$, one can use the Taylor expansion (Arfken, 1985) to obtain, as $\Delta\,T$ tends to zero,

$$\Delta\,R = \frac{\partial\,R}{\partial\,T}\Delta\,T + \frac{\partial\,R}{\partial\,\alpha}\frac{\partial\,\alpha}{\partial\,T}\Delta\,T + \mathcal{O}\left((\Delta\,T)^2\right). \qquad (1)$$

Here, $\mathcal{O}(x)$ is a function such that $\mathcal{O}(x) \le C\,x$ as soon as $0 < x < \epsilon$ for some positive constants C and ϵ.

Introducing the notations

$$\frac{1}{\lambda_0} = \frac{\partial\,R}{\partial\,T}, \text{ and } f = -\lambda_0\,\frac{\partial\,R}{\partial\,\alpha}\frac{\partial\,\alpha}{\partial\,T},$$

for the "reference sensitivity" λ_0 and the "feedback factor" f, we obtain

$$\Delta\,R = \frac{1-f}{\lambda_0}\,\Delta\,T + \mathcal{O}\left((\Delta\,T)^2\right), \qquad (2)$$

which readily leads to

$$\Delta\,T = \frac{\lambda_0}{1-f}\Delta\,R + \mathcal{O}\left((\Delta\,T)^2\right), \qquad (3)$$

as soon as $f \neq 1$.

Roe and Baker (2007) drop the higher-order terms in (3) to obtain their equation (S4):

$$\Delta\,T = \frac{\lambda_0}{1-f}\Delta\,R. \qquad (4)$$

This equation provides a basis for studying climate sensitivity and it leads directly to these authors' main conclusions.

A peculiar property of (4) is that its right-hand side (rhs) diverges as f approaches unity, which is used to advocate the possibility of intrinsically large climate-system sensitivity and hence the irreducible uncertainty of future-climate projections. Roughly speaking, Roe and Baker (2007) use the following argument: If the derivative of $R(T)$ with respect to T is close to 0, then the derivative of T with respect to R is very large, and a small change in the radiation R corresponds to a large change in the temperature T.

Such an argument, though, is only valid for an essentially linear dependence $R = R(T)$. In a generic nonlinear climate model, its validity will be limited to a small neighborhood of the point where the derivative of the function $R(T)$ is actually 0. We illustrate this crucial point in detail and give examples in Section 2 below.

It is worth noticing that, since one seeks the temperature change $\Delta\,T$ that results from a change $\Delta\,R$ in the forcing, it might be preferable to consider the inverse function $T = T(R)$ or, more precisely, $T = T(R, \alpha(R))$ and the corresponding Taylor expansions

$$\Delta\,T = \frac{\partial\,T}{\partial\,R}\Delta\,R + \mathcal{O}\left((\Delta\,R)^2\right),$$

$$\Delta\,T = \frac{\partial\,T}{\partial\,R}\Delta\,R + \frac{\partial\,T}{\partial\,\alpha}\frac{\partial\,\alpha}{\partial\,R}\Delta\,R + \mathcal{O}\left((\Delta\,R)^2\right).$$

The main conclusions of our analysis will not be affected by the particular choice of direct or inverse expansion, provided the nonlinearities are correctly taken into account.

2 A self-consistent sensitivity analysis

It is easily seen from the discussion in Section 1.2, especially from Eq. (2), that the relationship (4) is an *approximation* that is only valid subject to the assumptions that (*a*) the higher-order terms in the expansion of $\Delta\,T$ are vanishingly small:

$$\mathcal{O}\left((\Delta\,T)^2\right) \ll \frac{1-f}{\lambda_0}\Delta\,T;$$

and (*b*) the quantity in the rhs of this inequality is not itself equal to 0.

If one assumes, for instance, that $\mathcal{O}\left((\Delta\,T)^2\right) \sim C\,(\Delta\,T)^2$, where the precise meaning of $g(x) \sim f(x)$ is given by $\lim_{x\to 0} g(x)/f(x) = 1$, then the assumptions (*a*,*b*) above hold for $\Delta\,T$ that satisfy both of the following conditions

$$0 < \Delta\,T \ll (1-f)/(\lambda_0\,C) \quad \text{and} \quad 0 < \Delta\,T < \epsilon, \qquad (5)$$

where C and ϵ are defined after Eq. (1). The first of these conditions implies that the range of temperatures within which the aproximation (4) works vanishes as the feedback factor f approaches unity. Hence, all the results based on this approximation — including precisely the main conclusions of Roe and Baker (2007) — no longer apply outside a vanishingly small neighborhood of $f = 1.0$.

The asymptotic behavior we assumed above for $\mathcal{O}\left((\Delta T)^2\right)$ is not exotic. Consider for instance the function $R = T^2$ in the neighborhood of $R = 0$. Its Taylor expansion

$$\Delta R = 2\,T\Delta T + \mathcal{O}\left((\Delta T)^2\right)$$

can be used to obtain, ignoring the second-order term,

$$\Delta T \approx \Delta R/(2\,T). \tag{6}$$

The last equation would seem to imply that the growth of ΔT is inversely proportional to T itself, so the change in T should increase infinitely fast as T goes to 0, a rather annoying contradiction. The way out of this conundrum is to realize that the change ΔT given by Eq. (6) is only valid in a small vicinity of $T = 0$ and cannot be extrapolated to larger values. Of course, we all know that the function $R = T^2$ is nicely bounded and smooth in the vicinity of 0, but it is essential to take into account the second term in its Taylor expansion in this vicinity. We show in Section 3 below that this simple example depicts the essential dependence of the Earth surface temperature on the global solar radiative input.

In summary, the linear approximation of the function $R(T)$ derived by Roe and Baker (2007) from its Taylor expansion is not valid when f approaches unity. In this case — which is precisely the situation emphasized by these authors — the higher-order terms "hidden" inside $\mathcal{O}\left((\Delta T)^2\right)$, which they neglect, are indispensable for a correct, self-consistent climate sensitivity analysis.

A correct analysis of the case when f approaches unity needs to start with a Taylor expansion that keeps the second-order term

$$\Delta R = \frac{1-f}{\lambda_0}\Delta T + a\,(\Delta T)^2 + \mathcal{O}\left((\Delta T)^3\right),$$

where $a = \frac{1}{2}\partial^2 R/\partial T^2$. If $\mathcal{O}\left((\Delta T)^3\right)$ is much smaller than the other two terms on the rhs, then the temperature change can be approximated by a solution of the quadratic equation

$$\frac{1-f}{\lambda_0}\Delta T + a\,(\Delta T)^2 = \Delta R.$$

The real solutions to the latter equation, if they exist, are given by

$$\Delta T_{1,2} = \frac{1}{2}\left(\frac{f-1}{a\,\lambda_0} \pm \sqrt{\left(\frac{1-f}{a\,\lambda_0}\right)^2 + \frac{4\,\Delta R}{a}}\right).$$

In particular, when $f \approx 1$, then

$$\Delta T_{1,2} \approx \pm\sqrt{\frac{\Delta R}{a}}. \tag{7}$$

One can see from Eq. (7) that the proximity of the feedback factor f to unity no longer plays an important role in the qualitative behavior of the equilibrium temperature.

In general, one can consider an arbitrary number of terms in the Taylor expansion of $R(T)$. The very fact that one relies on the validity of the Taylor expansion implies that $R(T)$ is bounded and sufficiently smooth; in other words, a divergence of the equilibrium temperature due to a small change in the forcing contradicts the very assumptions on which Roe and Baker (2007) based their sensitivity analysis.

3 Sensitivity for energy balance models (EBMs)

3.1 Model formulation

We consider here a highly idealized type of model that connects the Earth's temperature field to the solar radiative flux. These models' central idea stems from the pioneering works of Budyko (1969) and Sellers (1969), and they have been subsequently generalized and used for many studies of climate stability and sensitivity, cf. Held and Suarez (1974); North (1975); Ghil (1976); North *et al.* (1981); and Ghil and Childress (1987), among others.

The key assumption is that the rate of change of the global average temperature T is determined only by the net balance between the absorbed radiation R_{i} and emitted radiation R_{o}:

$$c\frac{\mathrm{d}T}{\mathrm{d}t} = R_{\mathrm{i}}(T) - R_{\mathrm{o}}(T). \tag{8}$$

Many distinct versions of this type of model have been formulated and studied. The differences lie in the dependence of the temperature on one (latitude θ), two (latitude θ and longitude ϕ) or no (global value only) spatial variables, and in the dependence of the absorbed and emitted radiation (R_{i} and R_{o}) on the global temperature T, as in Eq. (8) above, or on the temperature field, $T(\theta)$ or $T(\theta,\phi)$. The resulting EBMs are called zero-, one- or two-dimensional (0-D, 1-D or 2-D), depending on their number of spatial variables.

For simplicity, we follow here the global, 0-D version of Crafoord and Källén (1978) and Ghil and Childress (1987), and let

$$R_{\mathrm{i}} = \mu\,Q_0\,(1-\alpha(T)), \quad R_{\mathrm{o}} = \sigma\,g(T)\,T^4. \tag{9}$$

In this formulation, the planetary *ice-albedo feedback* α decreases in an approximately linear fashion with T, within an intermediate range of temperatures, and is nearly constant for large and small T, while Q_0 is the reference value of the global mean solar radiative input, σ is the Stefan-Boltzmann constant, and $g(T)$ is the grayness of the Earth, *i.e.* its deviation from the black-body radiative emission σT^4. The parameter $\mu \approx 1.0$ multiplying Q_0 indicates by how much the global insolation deviates from its reference value.

We model the ice-albedo feedback by

$$\alpha(T;\kappa) = c_1 + c_2\,\frac{1-\tanh\left[\kappa\,(T-T_{\rm c})\right]}{2}. \qquad (10)$$

This parametrization represents a smooth interpolation between the piecewise-linear formula of Sellers-type models, like those of Ghil (1976) or Crafoord and Källén (1978), and the piecewise-constant formula of Budyko-type models, like those of Held and Suarez (1974) or North (1975); see also Table 10.1 of Ghil and Childress (1987).

Figure 1a shows four profiles of our ice-albedo feedback $\alpha(T)$ as a function of T, depending on the value of the steepness parameter κ. The profile for $\kappa \gg 1$ would correspond roughly to a Budyko-type model, in which the albedo α takes only two constant values, high and low, depending on whether $T < T_{\rm c}$ or $T > T_{\rm c}$. The other profiles, shown in the figure, for smaller values of κ, correspond to Sellers-type models, in which there exists a transition ramp between the high and low albedo values. Recall that the original Sellers (1969) profile — as used by Ghil (1976) and Crafoord and Källén (1978) in their analysis of 1-D and 0-D EBMs, respectively — uses a linear ramp between the high and low albedo values. Figure 1b shows the corresponding shapes of the radiative input $R_i(T)$.

The *greenhouse effect* is parametrized, as in Crafoord and Källén (1978) and Ghil and Childress (1987), by letting

$$g(T) = 1 - m\,\tanh\left((T/T_0)^6\right). \qquad (11)$$

Substituting this greenhouse effect parametrization and the one for the albedo into Eq. (8) leads to the following 0-D EBM:

$$\begin{aligned} c\dot{T} = {}& \mu\;\, Q_0\,(1-\alpha(T)) \\ & - \sigma T^4\left[1 - m\,\tanh\left((T/T_0)^6\right)\right], \end{aligned} \qquad (12)$$

where $\dot{T} = \mathrm{d}T/\mathrm{d}t$ denotes the derivative of global temperature T with respect to time t.

It is important to note that current concern, both scientific and public, is mostly with the greenhouse effect, rather than with actual changes in insolation. But in a simple EBM model — whether 0-D, 1-D or 2-D — increasing μ always results in a net increase in the radiation balance. It is thus convenient, and quite sufficient for the purpose at hand, to vary μ in the incoming radiation $R_{\rm i}$, rather than some other parameter in the outgoing radiation $R_{\rm o}$. We shall return to this point in Section 4.

3.2 Model parameters

The value S of the *solar constant*, which is the value of the solar flux normally incident at the top of the atmosphere along a straight line connecting the Earth and the Sun, is assumed here to be $S = 1370$ Wm^{-1}. The reference value of the global mean solar radiative input is $Q_0 = S/4 = 342.5$, with the factor 1/4 due to the Earth's sphericity.

The parameterization of the ice-albedo feedback in Eq. (10) assumes $T_c = 273$ K and $c_1 = 0.15$, $c_2 = 0.7$, which ensures that $\alpha(T)$ is bounded between 0.15 and 0.85, as in Ghil (1976); see Fig. 1. The greenhouse effect parametrization in Eq. (11) uses $m = 0.4$, which corresponds to 40% cloud cover, and $T_0^{-6} = 1.9 \times 10^{-15}$ K^{-6} (Sellers, 1969; Ghil, 1976). The Stefan-Boltzmann constant is $\sigma \approx 5.6697 \times 10^{-8}$Wm^{-2}K^{-4}.

3.3 Sensitivity and bifurcation analysis

3.3.1 Two types of sensitivity analysis

We distinguish here between two types of sensitivity analysis for the 0-D EBM (8). In the first type, we assume that the system is driven out of an equilibrium state $T = T_0$, for which $R_{\rm i}(T_0) - R_{\rm o}(T_0) = 0$, by an external force, and want to see whether and how it will return to a new equilibrium state, which may be different from the original one. This analysis refers to the "fast" dynamics of the system, and assumes that $R_{\rm i}(T) - R_{\rm o}(T) \neq 0$ for $T \neq T_0$; it is often referred to as *linear stability analysis*, since it considers mainly small displacements from equilibrium at $t = 0$, $T(0) = T_0 + T'(0)$, where $T'(0)$ is of order ϵ, with $0 < \epsilon \ll 1$, as defined in Section 2.

The second type of analysis refers to the system's "slow" dynamics. We are interested in how the system evolves along a *branch of equilibrium solutions* as the external force changes sufficiently slowly for the system to track an equilibrium state; hence, this second type of analysis always assumes that the solution is in equilibrium with the forcing: $R_\mathrm{i}(T) - R_\mathrm{o}(T) = 0$ for all T of interest. Typically, we want to know how sensitive the model solutions are to such a slow change in a given parameter, and so this type of analysis is called *sensitivity analysis*. In the problem at hand, we will study — again following Crafoord and Källén (1978) and Ghil and Childress (1987) — how changes in μ, and hence in the global insolation, affect the model's equilibria.

A remarkable property of the EBM governed by Eq. (8) is the existence of several stationary solutions that describe equilibrium climates of the Earth (Ghil and Childress, 1987). The existence and linear stability of these solutions result from a straightforward *bifurcation* analysis of the 0-D EBM (8), as well as of its 1-D counterparts (Ghil, 1976, 1994): there are two linearly stable solutions — one that corresponds to the present climate and one that corresponds to a much colder, "snowball Earth" (Hoffman *et al.*, 1981) — separated by an unstable one, which lies about 10 K below the present climate.

The existence of the three equilibria — two stable and one unstable — has been confirmed by such results being obtained by several distinct EBMs, of either Budyko- or Sellers-type (North *et al.*, 1981; Ghil, 1994). Stability to large perturbations in the initial state has been investigated by introducing a variational principle for 1-D EBMs of Sellers (Ghil, 1976) and of Budyko (North *et al.*, 1981) type, and it confirms the linear stability results.

3.3.2 Sensitivity analysis for a 0-D EBM

We analyze here the stability of the "slow," quasi-adiabatic (in the statistical-physics sense) dynamics of model (12). The energy-balance condition for steady-state solutions $R_\mathrm{i} = R_\mathrm{o}$ takes the form

$$\mu\, Q_0\,(1-\alpha(T)) = \sigma\, T^{\,4}\left[1 - m\,\tanh\left((T/T_0)^6\right)\right]. \quad (13)$$

We assume here, following the previously cited EBM work, that the main bifurcation parameter is μ; this happens to agree with the emphasis of Roe and Baker (2007) on climate sensitivity as the dependence of mean temperature T on global solar radiative input, denoted here by $Q = \mu Q_0$.

Figure 2 shows the absorbed and emitted radiative fluxes, R_i and R_o, as functions of temperature T for $\mu = 0.5, 1$ and 2.0. One can see that Eq. (13) may have one or three solutions depending on the value of μ: only the present, relatively warm climate for $\mu = 2.0$, only the "deep-freeze" climate for $\mu = 0.5$, and all three, including the intermediate, unstable one for present-day insolation values, $\mu = 1.0$. These steady-state climate values are shown as a function of the insolation parameter μ in the bifurcation diagram of Fig. 3.

The "fast" stability analysis (not presented here) shows that small deviations from an equilibrium solution, while all parameter values are kept fixed, may result in two types of dynamics, depending on the initial equilibrium: fast increase or fast decrease of the initial deviation (Ghil and Childress, 1987). The fast increase characterizes *unstable* equilibria: a small deviation $T'(0)$ from such an equilibrium T_0 forces the solution to go further and further away from the equilibrium. In practice, such equilibria will not be observed, since there are always small, random perturbations of the climate present in the system: just think of weather as representing such perturbations.

The fast decrease of the initial deviation $T'(0)$ characterizes *stable* solutions; only such equilibria can be observed in practice. The two stable solution branches of (12) are shown by solid lines in Fig. 3, while the unstable branch is shown by the dashed line. The arrows show the direction in which the temperature will change when drawn away from an equilibrium by external forces. This change, whether away from or towards the nearest equilibrium, is fast compared to the one that occurs along either solution branch (Ghil, 1976, 1994).

3.3.3 Bifurcation analysis

Given the choice of model parameters, the present climate state corresponds to the upper stable solution of Eq. (12), at $\mu = 1$ (see Fig. 3). It lies quite close to the bifurcation point $(\mu, T) \approx (0.9, 280\ \mathrm{K})$, where the stable and unstable solutions merge.

The so-called *normal form* of this bifurcation is given by the equation

$$\dot{X} = \bar{\mu} - X^2, \quad (14)$$

where X is a suitably normalized form of T, and $\bar{\mu}$ is a normalized form of μ. Equation (14) describes the dependence between T and μ in a small neighborhood of the bifurcation point. In particular, the stable equilibrium branch is described by

$$X = +\sqrt{\bar{\mu}};$$

this result has exactly the same form as the positive solution of Eq. (7), given by our self-consistent analysis of climate sensitivity in the presence of genuine nonlinearities, cf. Section 2. Hence, the derivative $\mathrm{d}X/\mathrm{d}\bar{\mu}$, and thus $\mathrm{d}T/\mathrm{d}\mu$, goes to infinity as the model approaches the bifurcation point; this is exactly the situation discussed earlier in Section 2.

It is important to realize that this parabolic form of temperature dependence on insolation change is not an accident due to the particularly simple form of EBMs. Wetherald and Manabe (1975) clearly showed, in a slightly simplified general circulation model (GCM), that not only the mass-weighted temperature of their total atmosphere, but also the area-weighted temperatures of each of their five model levels, exhibits such a parabolic dependence on fractional radiative input; see Fig. 5 in their paper. Moreover, these authors emphasize that "As stated in the Introduction, it is not, however, reasonable to conclude that the present results are more reliable than the results from the one-dimensional studies mentioned above simply because our model treats the effect of transport explicitly rather than by parameterization. [...] Nevertheless, it seems to be significant that both the one-dimensional and three-dimensional models yield qualitatively similar results in many respects."

In fact, rigorous mathematical results demonstrate that the *saddle-node bifurcation* whose normal form is given by Eq. (14) occurs in several systems of nonlinear PDEs, such as the Navier-Stokes equations (Constantin *et al.*, 1989; Temam, 1997). We emphasize, though, that this does not cause the temperature to increase rapidly due to small changes in insolation: the presence of the bifurcation point will result in small, positive changes of global temperature for slow, positive changes in μ, while it may throw the climate system into the deep-freeze state for slow, negative changes in μ.

4 Discussion

4.1 How sensitive is climate?

Making projections of climate change for the next decades and centuries, evaluating the human influence on future Earth temperatures, and making normative decisions about current and future anthropogenic impacts on climate are key tasks that all require solid scientific expertise, as well as responsible moral reasoning. Well-founded approaches to handle the moral aspects of the problem are still being debated [*e.g.*, Hillerbrand and Ghil (2008)]. It is that much more important to master existing tools for acquiring accurate and reliable scientific evidence from the available data and models. Several of these tools come from the realm of nonlinear and complex dynamical systems (Lorenz, 1963; Smale, 1967; Ghil and Childress, 1987; Ghil, 1994).

A straightforward analysis, carried out in Section 2 of this paper, shows that a proper treatment of the higher-order terms in a climate model with multiple feedbacks does not reveal the exaggerated sensitivity to forcing that was used by Roe and Baker (2007) to advocate intrinsic unpredictability of climate projections. Hence, while the general *human* concern about climate sensitivity expressed by these authors should be reasonably shared by many, their *scientific* conclusions cannot be supported by the linear model considered in their paper.

Accordingly, conclusions about the likelihood of extreme warming resulting from small changes in anthropogenic forcing can hardly be used to support political proposals [*e.g.*, Allen and Frame (2007)] that claim to provide future directions for the climate-related sciences. It seems to us that Roe and Baker's title question "Why Is Climate Sensitivity So Unpredictable?" remains wide open.

4.2 Is warm better than cool?

The S-shaped diagram of Fig. 3 — see also Fig. 10.6 in Ghil and Childress (1987) and Fig. 4 in Ghil (1994) — was used here to show the smoothness and boundedness of temperature changes as a function of insolation changes, away from a saddle-node bifurcation, like that of Eq. (7) in Section 2 or of Eq. (14) in Section 3.3.3. This S-shaped curve carries nevertheless a troublesome message: If the parameter μ were to slightly decrease — rather than increase, as it seems to have done since the mid-1970s, in the sense described in the last paragraph of Section 3.1 — then the climate system would be pushed past the bifurcation point at $\mu \approx 0.9$, and thus literally hover over an abyss. Indeed, the only way for the global temperature to go would be down, all the way to a deep-freeze Earth, with much lower temperatures than those of recent, Quaternary ice ages.

In the EBM context of Fig. 3, it would require an enormous, almost twofold increase in the insolation in order for a deep–freeze-type equilibrium to reach the bifurcation point at $\mu \approx 1.85$ and jump from there to $T \approx 350$ K, a temperature that sounds equally unpleasant. Within the broader context of the recent debates on how to exit a snowball-Earth state, very large, and possibly implausible increases in CO_2 levels would be required (Pierrehumbert, 2004).

To conclude — if the model we studied here has anything to do with reality, and substantial theoretical and GCM-simulation work (Wetherald and Manabe, 1975) suggests that it does — a relatively slight decrease in net radiation balance will lead to catastrophic consequences, while a slight increase in this balance sounds like a fairly mild way out of harm's way.

In it is worth recalling, in fact, that scientific and public concern about anthropogenic effects on climate started in the early 1970s, when temperatures, especially in the Northern Hemisphere, had been decreasing since the 1940s (SMIC, 1971; Gates and Mintz, 1975). In the light of recent discussions about geoengineering (Crutzen, 2006; MacCracken, 2006), and of still prevailing uncertainties about the relative effects of aerosols towards cooling vs. greenhouse gases towards warming (Ramanathan *et al.*, 2001), a slight, but well controlled shift towards a more positive radiation balance might have been a reasonable move for humankind to execute over the last couple of decades. This possibility leads directly to the next question.

4.3 How close are we to the bifurcation point?

Let us assume for the moment that the dangers of further warming will lead humanity to actually stop, and possibly reverse, the current trend of increasingly positive net radiation balance. Given, on the other hand, the dangers of another snowball Earth, we need to estimate the closeness of the climate system to the bifurcation point. The GCM simulations of Wetherald and Manabe (1975) (see again their Fig. 5) suggest that this point might lie no farther than -5% of the current solar constant.

Figure 4 here shows stable and unstable equilibrium solutions for different forms of the ice-albedo feedback, $\alpha = \alpha(T;\kappa)$; this form is determined by the value of the steepness parameter κ (cf. Fig. 1). The figure suggests that the steeper the ramp of the ice-albedo feedback function, *i.e.* the larger κ, the further away the bifurcation might lie. It also shows that for a very smooth dependence of the albedo on temperature, *i.e.* for a very small κ, there is no bifurcation at all: very small values of κ produce only a single-valued, smoothly increasing, stable equilibrium solution to (12) for any value of μ.

More generally, it seems worthwhile to carry out systematic bifurcation studies with atmospheric, oceanic and coupled GCMs to examine this question more carefully. Such studies are made possible by current computing capabilities and well-developed methods of numerical bifurcation theory (Dijkstra and Ghil, 2005; Simonnet *et al.*, 2009). This approach holds some promise in evaluating the distance of the current climate state from a catastrophic cooling, if and when we might get rid of a, more or less, catastrophic warming.

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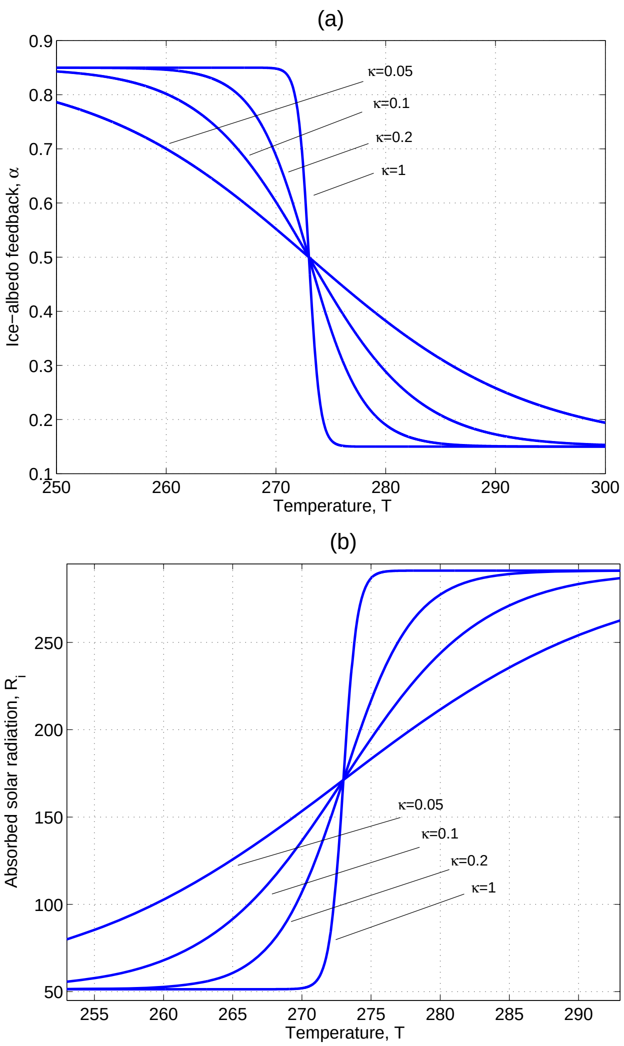

Fig. 1. Dependence of the absorbed incoming radiation R_i on the steepness parameter κ: (a) ice-albedo feedback $\alpha(T)$; and (b) absorbed radiation $R_\mathrm{i}(T)$, for different values of κ; see Eqs. (9) and (10).

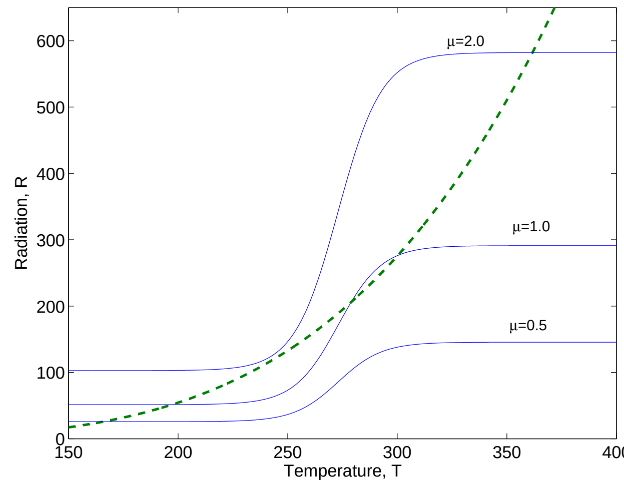

Fig. 2. Outgoing radiation R_o (dashed line) and absorbed incoming solar radiation R_i (solid lines) for our 0-D energy-balance model (EBM), governed by Eq. (12). The absorbed radiation is shown for $\mu = 0.5, 1$ and 2.0 (from bottom to top), while $\kappa = 0.05$. Notice the existence of one or three intersection points between the R_o curve and one of the R_i curves, depending on the value of μ; these points correspond to the equilibrium solutions of (12).

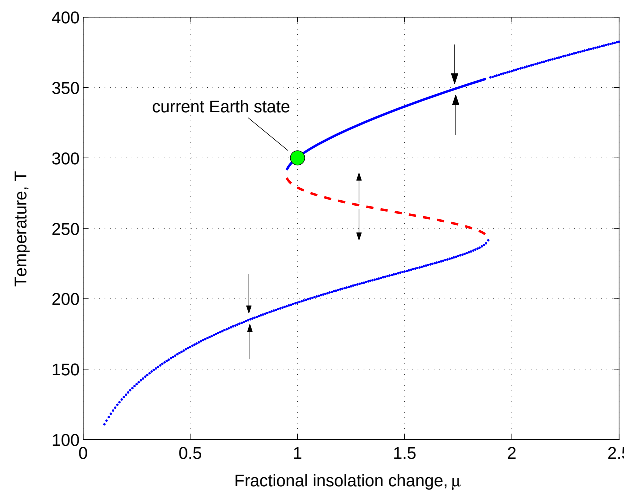

Fig. 3. Equilibrium solutions of the EBM (12) depending on the fractional change in insolation (μ). Notice the existence of two stable (solid lines) and one unstable (dashed line) solution branches. The arrows show the direction in which the global temperature will change after being displaced from a nearby equilibrium state by external forces. The current Earth state corresponds to the upper stable solution at $\mu = 1$; in this figure $\kappa = 0.05$.

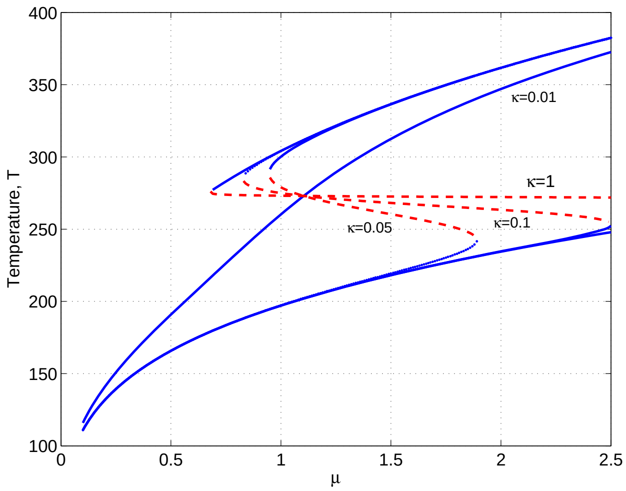

Fig. 4. Equilibrium solutions of the Energy Balance Model (12) for different ice-albedo feedback functions $\alpha(T,\kappa)$, for $\kappa = 0.01, 0.05, 0.1$ and 1. Notice existence of stable (solid lines) and unstable (dashed lines) solutions.