Random Dynamical Systems and Stochastic Structural Stability

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1. Can we reduce uncertainties in climate projections?

- Last IPCC report (AR4): GCM overfitting over 20th century leads to divergent behavior for the latter part of the 21st century.

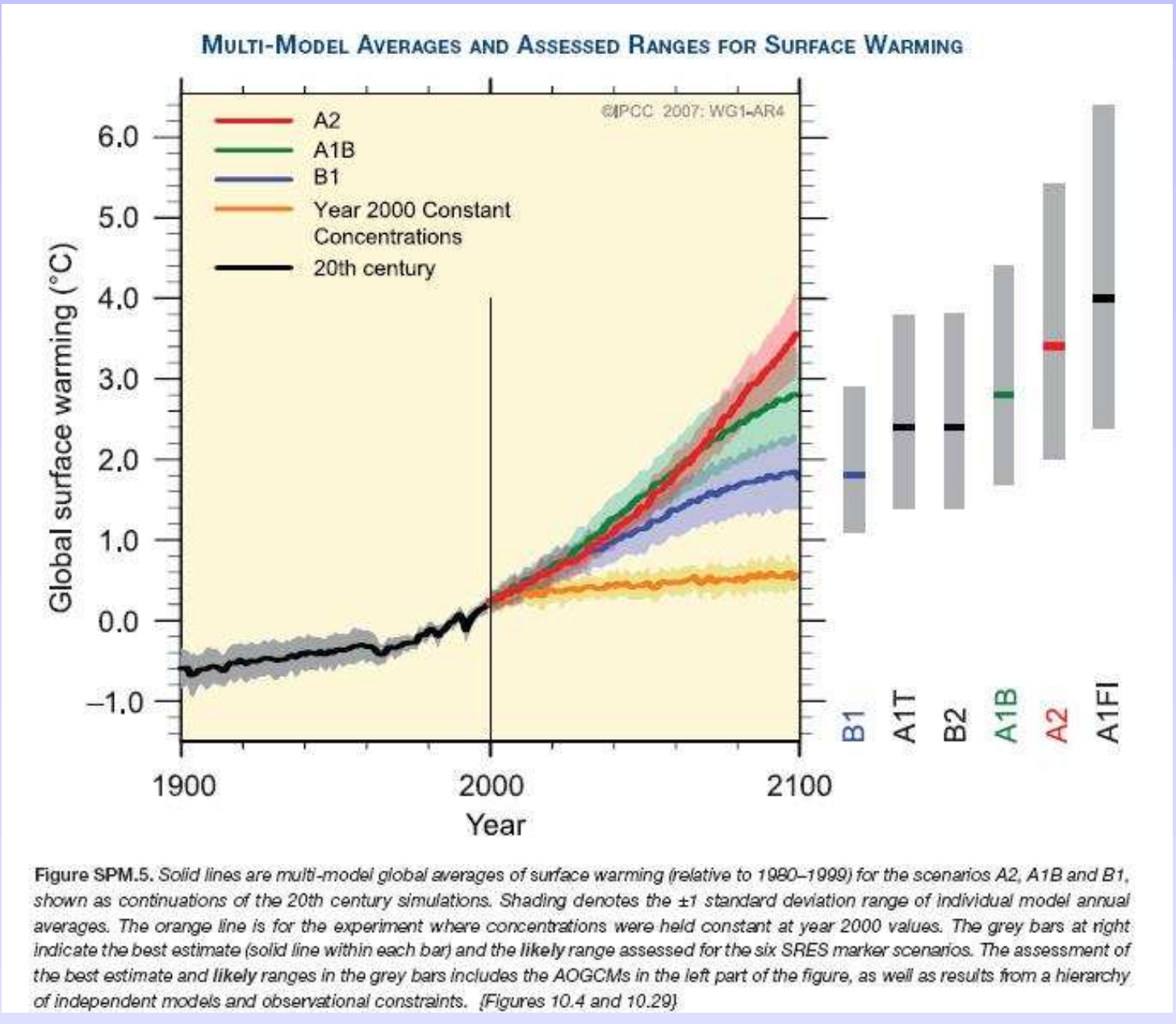
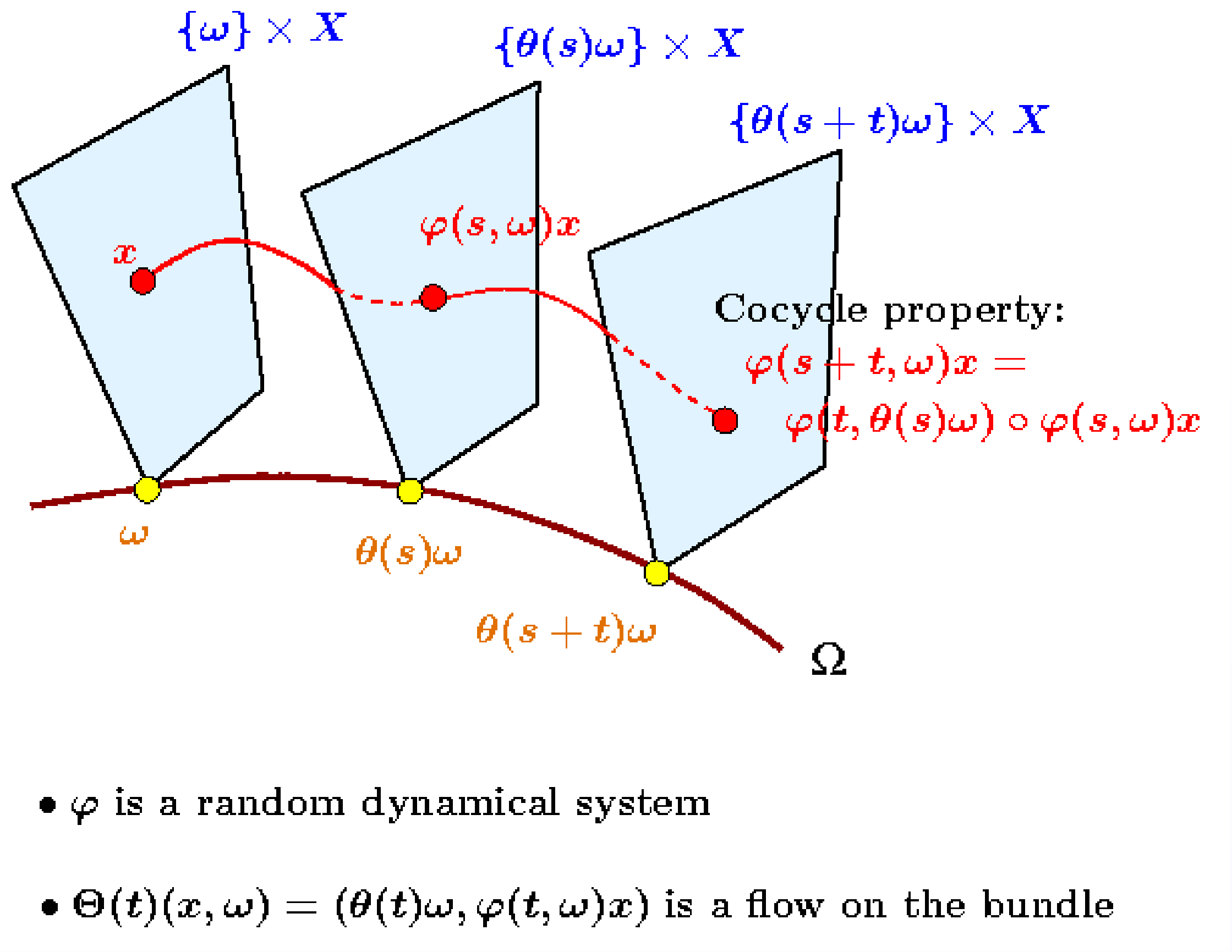


Figure SPM.5. *Solid lines are multi-model global averages of surface warming (relative to 1980–1999) for the scenarios A2, A1B and B1, shown as continuations of the 20th century simulations. Shading denotes the ±1 standard deviation range of individual model annual averages. The orange line is for the experiment where concentrations were held constant at year 2000 values. The grey bars at right indicate the best estimate (solid line within each bar) and the* ***likely*** *range assessed for the six SRES marker scenarios. The assessment of the best estimate and* ***likely*** *ranges in the grey bars includes the AOGCMs in the left part of the figure, as well as results from a hierarchy of independent models and observational constraints. {Figures 10.4 and 10.29}*

- This problem may be due to the fact that GCMs are structurally unstable, in the sense of deterministic dynamical systems.
- In fact, the climate system contains unresolved, random components ⇒ need for stochastic parametrizations in GCMs and hence a new concept of structural stability ⇒ Random Dynamical Systems.

2. What are Random Dynamical Systems (RDS)?

- RDS theory is a combination of measure and dynamical systems theory: "Bremen group" — L. Arnold (1998).
- RDS provide a geometric description of random systems: Flows on the bundle "Probability space" × "Phase space" X.

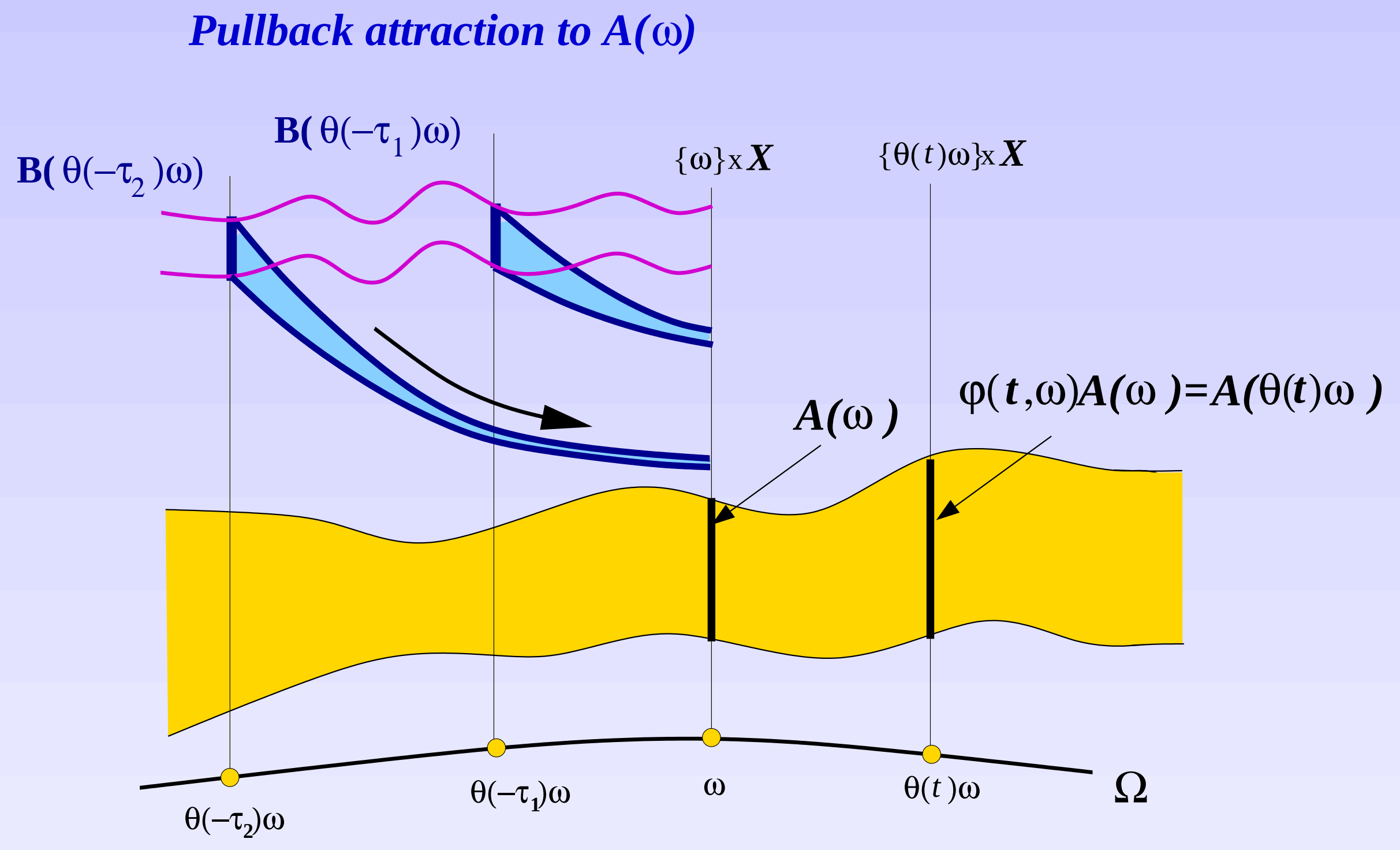


- φ is a random dynamical system
- $\Theta(t)(x,\omega) = (\theta(t)\omega, \varphi(t,\omega)x)$ is a flow on the bundle

- $\theta(t)$ is a model of the noise; it is called the driving system and must preserve the statistics!

3. What we would like to do first

- We want to compare the dynamical behavior of several GCMs in a "smart way."
- Comparing merely the probability density functions (PDFs) is not sufficient to understand the differences in underlying dynamics.
- We use the concept of **random attractor** $\mathcal{A}(\omega)$ and **stochastic topological conjugacy**:



- **Stochastic conjugacy**: two cocyles $\varphi_1(t,\omega)$ and $\varphi_2(t,\omega)$ are conjugated iff there exists a **random homeomorphism** h (i.e., a random, but smooth change of variables) such that:

$$\varphi_1(t,\omega) = h(\theta(t)\omega)^{-1} \circ \varphi_2(t,\omega) \circ h(\omega)$$

- We want to classify GCM behavior using these RDS tools and to study this classification as a function of the noise intensity. To do so, we consider "toy" models first.

4. V.I. Arnol'd's family of diffeomorphisms of the circle

- We want to perform a classification in terms of **stochastic equivalence**.

$$x_{n+1} = F_{\Omega,\epsilon}(x_n) := x_n + \Omega - \epsilon \sin(2\pi x_n) \bmod 1$$

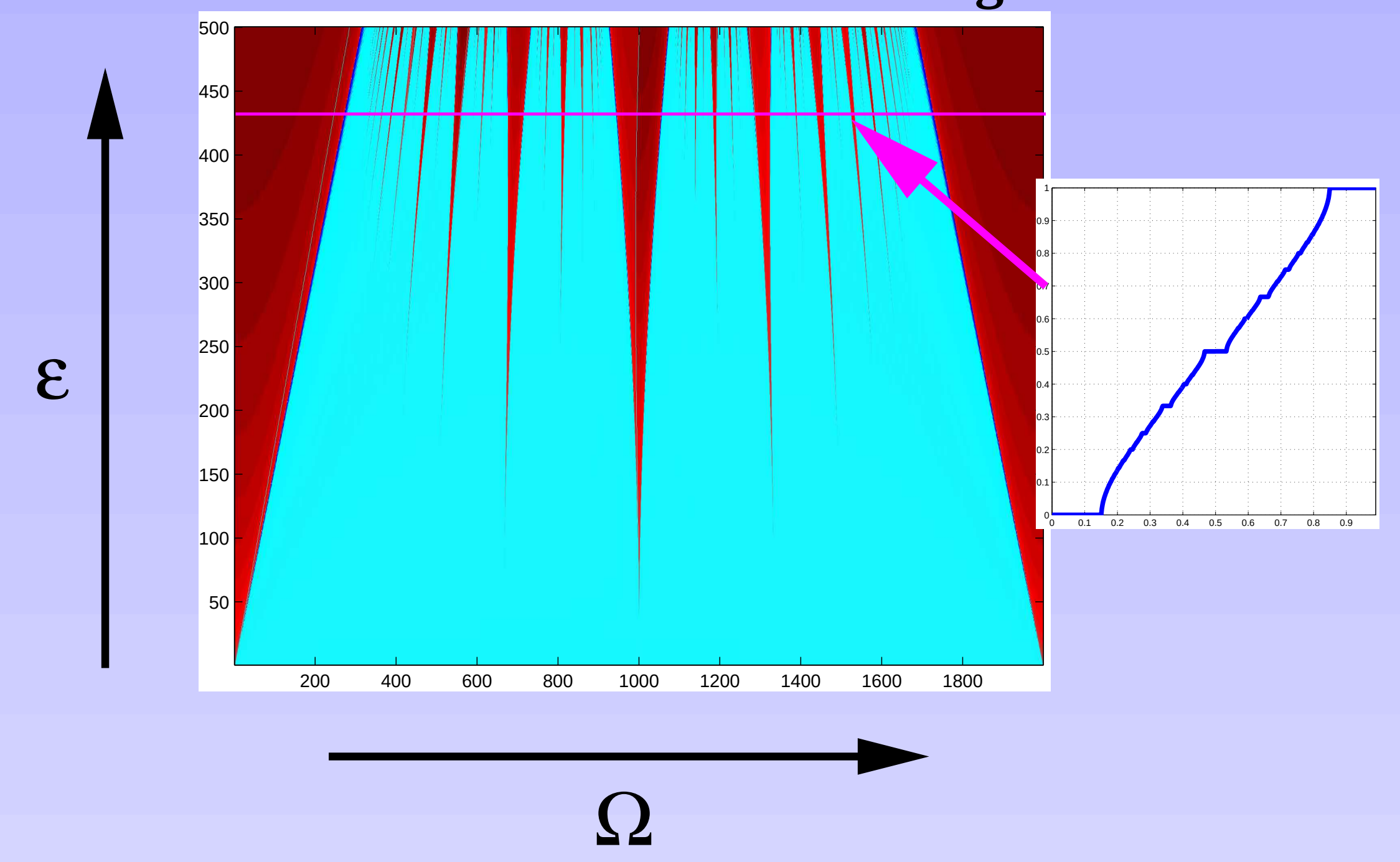
- Frequency-locking phenomena and the Devil's staircase
- Topological classification of the Arnol'd family, in parameter space:

(i) countable regions of structural stability *vs.*
(ii) uncountable structurally unstable systems with nonzero Lebesgue measure.

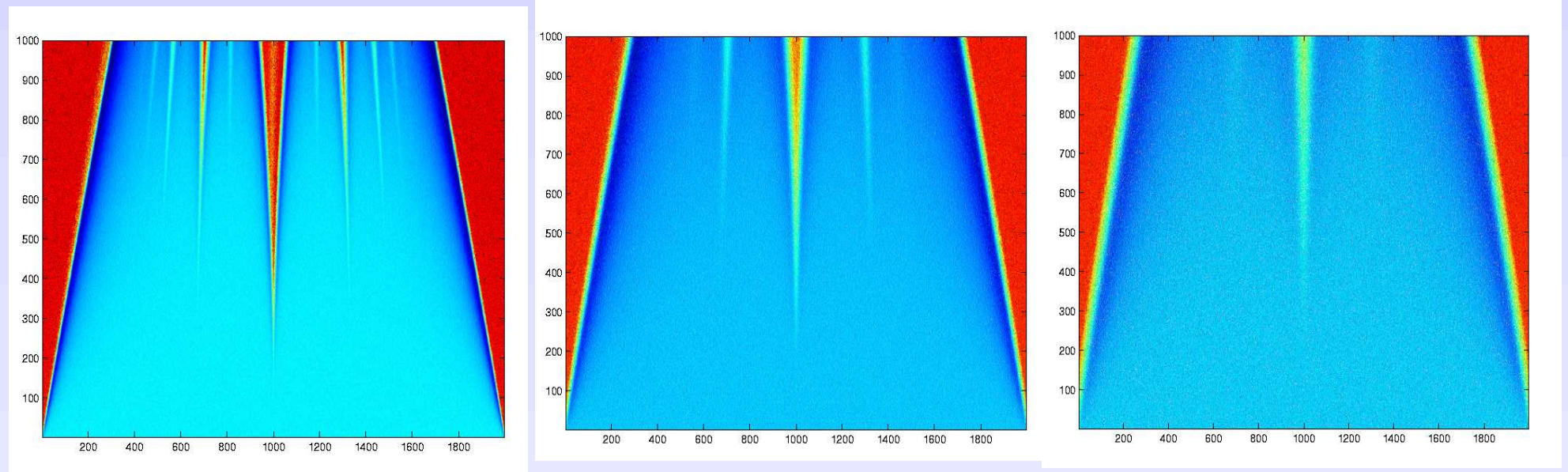
- Two types of attractors: periodic orbits *vs.* the whole circle S^1 (see next section).

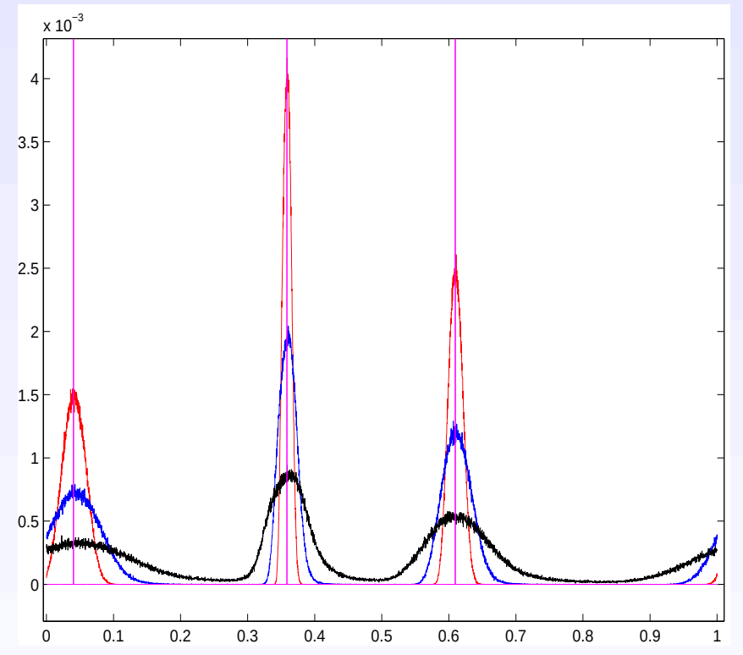
5. Effect of the noise on the Arnol'd family

Deterministic Arnol'd tongues



Effect of the noise (additive, amplitude σ).





Effect of the noise on the PDF of Arnol'd's tongue 1/3

- The number of classes becomes **FINITE**: only the most robust tongues survive!
- Irrational rotations becomes all stochastically equivalent to $\mathcal{A}(\omega) = \{a(\omega)\}$, where a is a random point.
- The noise has a dramatic impact on the sensitivity to the initial state.

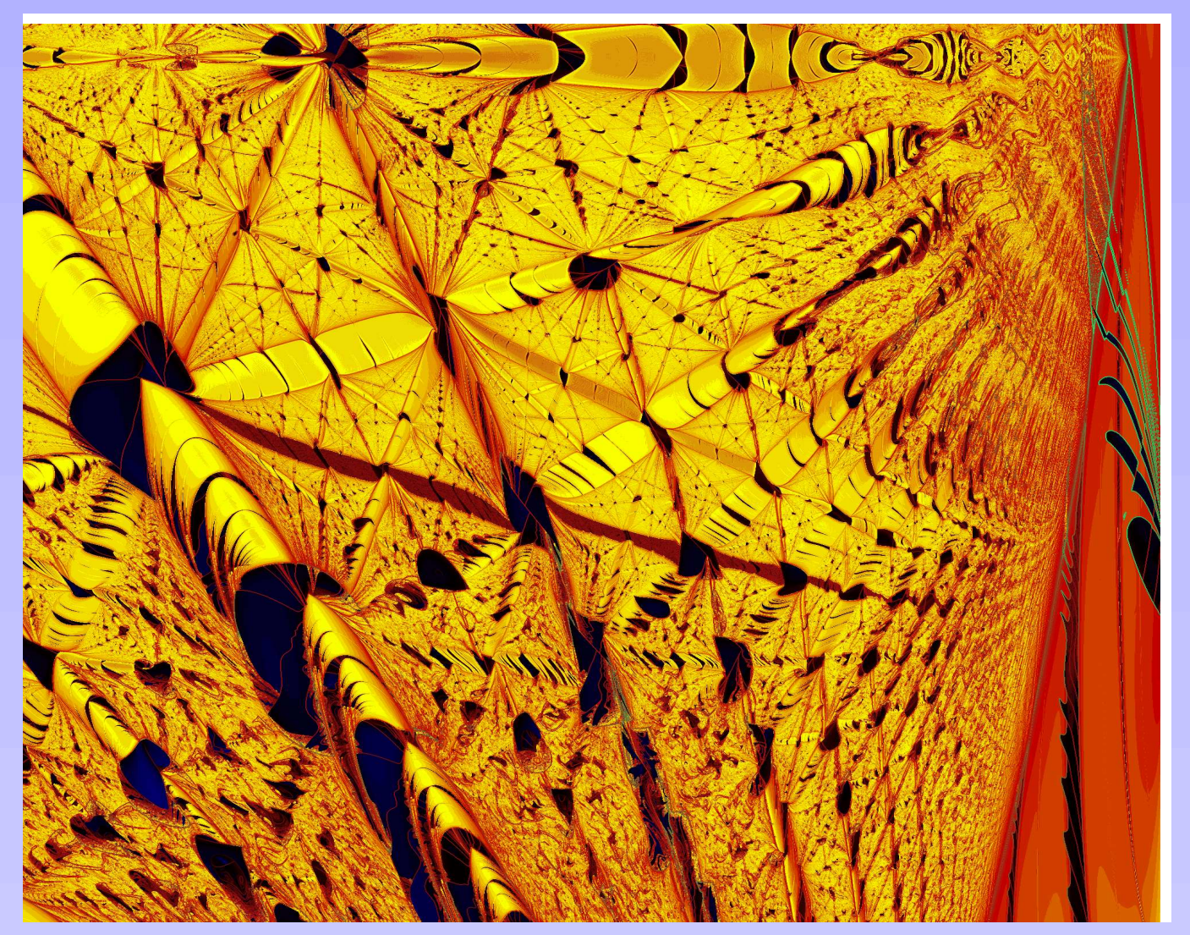
6. The torus map: a paradigm of complex systems

We extend the previous toy model to a more complex one:

$$\begin{aligned} x_{n+1} &= x_n + \Omega_1 - \epsilon \sin(2\pi y_n) \bmod 1 \\ y_{n+1} &= y_n + \Omega_2 - \epsilon \sin(2\pi x_n) \bmod 1 \end{aligned}$$

- **Web of resonances** (partial and full) + **chaos** !
- This provides a more realistic paradigm of observed dynamics.

Deterministic web of resonances



Effect of the noise (additive, amplitude $\sigma = 0.15$).

- The web of resonances is nonlinearly modified. This smoothing is linked with stochastic normal form theory.

7. Concluding remarks

- **Collapse** of the attractor's dimension: $\lim_{\sigma \to 0} \dim \mathcal{A}_\sigma(\omega) < \dim \mathcal{A}_0$ as the noise intensity σ goes to zero — is this always the case?
- Stochastic parametrization ⇒ **gain of structural stability** for random attractors.
- These results hold for pertinent deterministic models that are stochastically perturbed.
- RDS theory offers a meaningful framework for classifying rich behavior in stochastic modeling.

References

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