Perturbations of Plane Couette Flow in Stratified Fluid and Origin of Cloud Streets

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Small perturbations of plane Couette flow in stably and unstably stratified fluid are considered. It is found that the system is more unstable when it is bounded both above and below than when its depth is infinite, but a finite negative Richardson number J is required to maintain the perturbation for both cases. For the former case, this limiting Richardson number is $-3k_1^2/4(k_1^2 + k_2^2)$, while for the latter it is $-2k_1^2/(k_1^2 + k_1^2)$, where k_1 and k_2 are wavenumbers in the mean flow and the transversal direction. These results show that in an unstably stratified layer of Couette flow, the preferred mode of motion is roll-type convection, with the dimension in the wind direction much larger than in the transversal direction. The amplification factor σ for the perturbations has been determined as a function of the modified Richardson number $\bar{J} = gS_Z U_z^{-2}(1 + k^{-}{}_1^2 k_2^2)$, and the dimensionless wavenumber $\alpha = h(k_1^2 + k_1^2)^{\frac{1}{2}}$. Four different regimes have been found, each corresponding to a different type of perturbation. An application of the theory is made to the formation of longitudinal cloud rolls observed in the earth's atmosphere and in certain laboratory experiments.

I. INTRODUCTION

It has been known from laboratory experiments[1] and also from cloud observations in the earth's atmosphere[2–4] that when a mean flow with vertical shear is present in an unstably stratified fluid layer, convection tends to take the form of long rolls, with their axes parallel to the mean flow. The reason for such cloud formation has been attributed previously by the author[5] to the prohibitive effect of the shear flow for windward wave motion without proof. In this paper we investigate this problem in more detail, by examining the properties of the perturbations superimposed on a shearing flow in a stratified fluid and determining the most preferred mode of motion under various combinations of shear and stratification.

The stability problem of an unbounded layer of stably stratified fluid with constant shear has been treated by Case[6] and Dyson,[7] with the result that the system is stable for all perturbations when the Richardson number J is greater than $\frac{1}{4}$, while for $0 < J < \frac{1}{4}$ the system is neither exponentially unstable nor neutral, but the initial density perturbation behaves as $t^{-0.5+(0.25-J)^{\frac{1}{2}}}$ for large t.

While the results obtained by Case and Dyson are very interesting indeed, their model makes U unbounded and is therefore somewhat unrealistic. It seems very desirable to make a similar analysis for the system which is bounded both above and below. In fact, the stability character of such a system has been treated by Hoiland,[8] and Eliassen, Hoiland, and Riis,[9] who also included unstable stratification in their investigations. However, Hoiland treated only the singular perturbation of wavenumber zero, while Eliassen, Hoiland, and Riis have taken for their treatment the other extreme, the perturbation with an infinitely large wavenumber k_1. Further, their result concerning the asymptotic behavior for large t when eigensolutions do not exist is somewhat conflicting, while their interesting statement on the distribution of eigenvalues for $J > \frac{1}{4}$ is given without proof. Since this case is of considerable interest to us, especially in relation to the decay of turbulence in a stably stratified fluid, we shall also include stable stratification in our study and try to make a complete mapping of the behavior of the perturbations for both positive and negative values of J.

We mention that, because of our attempt to make a detailed analysis of the behavior of the perturbations in a flow of constant shear in the entire layer, the problem of the influence of a variable vertical shear will not be discussed in this paper. Such problems have been treated previously by Taylor,[10]

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[1] D. Avsec, Sci. and Tech. Publ. of Air Ministry, No. 155, Paris (1939).
[2] J. Kuettner, Tellus **11**, 267 (1959).
[3] J. Winston, Monthly Weather Rev. **88**, 295 (1960).
[4] J. S. Malkus and C. Ronne, Am. Geophys. Union, Monograph No. **5**, 45 (1960).
[5] H. L. Kuo, Tellus **13**, 441 (1961).
[6] K. M. Case, Phys. Fluids **3**, 144 (1960).
[7] F. J. Dyson, Phys. Fluids **3**, 155 (1960).
[8] E. Hoiland, Geofys. Publikasjoner, Norske Videnskaps-Akad. Oslo **18**, No. 10, 1 (1953).
[9] A. Eliassen, E. Hoiland, and E. Riis, Institute of Weather and Research, Norwegian Acad. Sci. Letters. Publ. No. 1, 1 (1953).
[10] G. I. Taylor, Proc. Roy. Soc. (London) **A132**, 499 (1931).

Goldstein,[11] and Drazin,[12] but an extension to the convection problem in such a shear flow awaits a further investigation.

II. GOVERNING EQUATIONS AND PROPERTIES OF SIMPLE PERTURBATIONS

The static state of the atmosphere is supposed to be characterized by a mean motion U, a mean pressure p_0, and a mean density ρ_0, which are assumed to be functions of the vertical coordinate z only. A small perturbation is superimposed on this basic state so that the actual pressure, density and the velocity components at a point (x, y, z) are $p_0 + p$, $\rho_0 + \rho$, and $U + u$, v, w. The atmosphere is considered as compressible and inhomogeneous; therefore it is convenient to introduce $s = \log(\theta/\theta_0)$ as another perturbation variable, where θ is the actual (disturbed) potential temperature and θ_0 is its value in the undisturbed state.

In this investigation we shall be concerned mainly with the stability of the perturbations under the combined influences of the shearing mean flow and a stable or an unstable stratification. Such problems can conveniently be treated under the inviscid and adiabatic approximations. However, in order to fix the preferred scale of motion properly in the *cross-wind direction* when the stratification is *unstable*, it is necessary to take into consideration the dissipative effects of friction and heat conduction. For simplicity, we assume that these effects are proportional to the sum of the squares of the horizontal wavenumbers and the inverse square of a representative depth d_1 of the convective layer, and that the two coefficients are equal. In this way we wish to include the essential part of the influence of internal friction and some boundary effect without solving the higher-order equations, even though it may not give very accurate results in dealing with problems where the boundary layer is of primary importance. The linearized equations of motion and the first law of thermodynamics are then given by

$$Lu + U_z w = -(\partial/\partial x)(p/\rho_0), \quad (2.1)$$

$$Lv = -(\partial/\partial y)(p/\rho_0), \quad (2.2)$$

$$Lw = -(\partial/\partial z)(p/\rho_0) + gs + S_Z p/\rho_0, \quad (2.3)$$

$$Ls + S_Z w = 0, \quad (2.4)$$

where

$$L \equiv \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + \nu\left(k_1^2 + k_2^2 + \frac{\pi^2}{d_1^2}\right),$$

d_1 is an effective depth, k_1 and k_2 being the wavenumbers in the x and the y directions,

$$S_Z = \theta_0^{-1}\,\partial\theta_0/\partial z = g/c_p T_0 + T_0^{-1}\,\partial T_0/\partial z,$$

T_0 being the undisturbed absolute temperature, and the subscripts denote partial differentiations.

Excluding sound waves and high-frequency gravitational waves from the present investigation, we may then neglect the local variation of density from the continuity equation and obtain

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \beta w \quad (2.5)$$

$$= 0 \text{ with Boussinesq approximation.}$$

Here $\beta = -\rho_0^{-1}\,\partial\rho_0/\partial z$ is the mean density stratification factor. It can easily be shown by making use of the ideal gas equation and the hydrostatic relation for the basic state that β and S_Z are connected by

$$\beta = (c_v/c_p R T_0) + S_Z. \quad (2.6)$$

Since β and S_Z involve T_0, they usually vary with height. However, as the percentage variation of T_0 is ordinarily very small, it may be taken as constant when not being differentiated, and therefore β can also be treated as a constant, without restricting the atmosphere to be isothermal.

For earth's atmosphere, we have $\beta \simeq 1 \times 10^{-4}\ \mathrm{m}^{-1}$ while the value of S_Z is usually about $10^{-5}\ \mathrm{m}^{-1}$ or less; we may therefore neglect S_Z against β. Within this degree of accuracy, the last term on the right side of Eq. (2.3) may also be neglected in comparison with the first.

Differentiating (2.1) with respect to x and (2.2) with respect to y and adding, and making use of (2.5), we obtain

$$L(w_z - \beta w) - U_z w_x = \nabla_1^2\, p/\rho_0, \quad (2.7)$$

where $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Differentiating (2.7) with respect to z and applying ∇_1^2 to (2.3) and combining the two resulting equations we obtain

$$L(\nabla^2 w - \beta w_z) - (U_{zz} + \beta U_z)w_x = g\nabla_1^2 s, \quad (2.8)$$

where $\nabla^2 = \nabla_1^2 + \partial^2/\partial z^2$.

Applying L to (2.8) and ∇_1^2 to (2.4) and combining the resulting equations we finally obtain the following equation in w:

$$L^2(\nabla^2 w - \beta w_z) - (U_{zz} + \beta U_z)Lw_x + gS_Z\nabla_1^2 w = 0. \quad (2.9)$$

[11] S. Goldstein, Proc. Roy. Soc. (London) **A132**, 524 (1931).
[12] P. G. Drazin, J. Fluid Mech. **4**, 214 (1958).

Notice that this equation differs from that for an incompressible heterogeneous liquid in two respects, viz., (i) the factor S_Z in the last term will be replaced by β and (ii), the term βU_z will be replaced by $-\beta U_z$ for incompressible flow in a heterogeneous fluid.

Introducing Fourier transforms with respect to x and y and a Laplace transform with respect to t for the dependent variables, we obtain

$$w(x, y, z, t) = \frac{1}{4\pi^2} \iint w_{k_1,k_2}(z, t) e^{-i(k_1x+k_2y)}\, dk_1\, dk_2,$$

$$w_{k_1,k_2,p}(z) = \int_0^\infty w_{k_1,k_2}(z, t) e^{-pt}\, dt, \tag{2.10}$$

where p may be complex, that is, $p = p_r + ip_i$. Substituting (2.10) into (2.9) we find that the quantity $W(z) \equiv e^{-\frac{1}{2}\beta z} w_{k_1,k_2,p}(z)$ satisfies the following equation:

$$\frac{d^2W}{dz^2} - \left[k^2 + \frac{\beta^2}{4} + \frac{U_{zz} + \beta U_z}{p_1 + ik_1U} ik_1 + \frac{gS_Zk^2}{(p_1 + ik_1U)^2}\right]W = \frac{1}{p_1 + ik_1U}\left[\frac{d^2W_0}{dz^2} - \left(k^2 + \frac{\beta^2}{4}\right)W_0\right] - \frac{gk^2e^{-\frac{1}{2}\beta z}}{(p_1 + ik_1U)^2} s_0, \tag{2.11}$$

where

$$p_1 = p + \nu(k_1^2 + k_2^2 + \pi^2/d_1^2),$$

$k^2 = k_1^2 + k_2^2$, and W_0 and s_0 denote the initial values of W and s.

Since $e^{-\beta z}$ is proportional to ρ_0, the appropriate boundary conditions are the vanishing of W at fixed horizontal boundaries and W remains finite or approaches zero as z approaches infinity.

For the stability problems that permit eigensolutions, we may set W_0 and s_0 to zero and look for eigensolutions for the equation

$$\frac{d^2W}{dz^2} - \left[k^2 + \frac{\beta^2}{4} + \frac{U_{zz} + \beta U_z}{U - C} - \frac{g\bar{S}_Z}{(U - C)^2}\right]W = 0, \tag{2.12}$$

where $\bar{S}_Z = S_Z(1 + k_2^2k_1^{-2})$ and $C = ip_1k_1^{-1}$. The real part, C_r, then gives the phase velocity of the normal mode perturbation in the x direction, while

$$p_r = k_1C_i - \nu(k^2 + \pi^2/d_1^2)$$

represents the exponential growth rate of this normal mode when the dissipative terms are taken into consideration.

Two conclusions can immediately be drawn from this equation about the relative stability of two-dimensional ($k_2 = 0$) and three-dimensional ($k_2 \neq 0$) perturbations, namely:

(i) For stably stratified fluid ($S_Z > 0$) the two-dimensional perturbation is more unstable than the three-dimensional perturbation of the same k_1, because the stabilizing effect of the positive S_Z is minimized in two-dimensional motion.

(ii) For unstably stratified fluid ($S_Z < 0$) the three-dimensional perturbation is more unstable than the two-dimensional perturbation, because the destablilizing effect of the negative S_Z is maximized by the former.

Thus for treating the problem of convection and also internal gravity waves in a shearing flow we must retain the cross-wind wavenumber k_2 in the perturbation.

Before discussing the perturbations in a shearing mean current, it is instructive to examine the properties of the perturbations in a constant mean flow $\bar{U}$. For this case (2.12) reduces to

$$\frac{d^2W}{dz^2} - \left[k^2 + \frac{\beta^2}{4} - \frac{gS_Zk^2}{(\bar{U} - C)^2k_1^2}\right]W = 0. \tag{2.13}$$

The solution of this equation may be written as

$$W_n = A_ne^{inz}, \tag{2.14}$$

provided C is given by

$$C \equiv C_x = \bar{U} \pm \frac{k}{k_1}\left(\frac{gS_Z}{k^2 + n^2 + \frac{1}{4}\beta^2}\right)^{\frac{1}{2}}. \tag{2.15}$$

If no specific boundary conditions for w is stipulated, the vertical wavenumber n is then free to take any values. On the other hand, if w is required to vanish at the levels $z = 0$ and $z = d_1$, then we must use the imaginary part of (2.14), i.e., use sin nz, only, and limit nd_1/π to nonzero integers. In either case, arbitrary initial perturbations can be represented by a summation of (2.14) over the three wavenumbers k_1, k_2, and n and therefore this set of eigensolutions is complete.

When S_Z is positive, (2.14) gives all the possible neutral internal gravity waves, and the second term on the right-hand side of (2.15) gives the phase velocity in x direction relative to the mean current $\bar{U}$. Since $\omega = k_1C$ is the frequency of the normal mode perturbation, the phase velocities in the y and z directions are given by

$$C_y \equiv \frac{\omega}{k_2} = \frac{\bar{U}k_1}{k_2} \pm \frac{k}{k_2}\left(\frac{gS_Z}{k^2 + n^2 + \frac{1}{4}\beta^2}\right)^{\frac{1}{2}},$$
$$C_z \equiv \frac{\omega}{n} = \frac{\bar{U}k_1}{n} \pm \frac{k}{n}\left(\frac{gS_Z}{k^2 + n^2 + \frac{1}{4}\beta^2}\right)^{\frac{1}{2}}. \qquad (2.15a)$$

Equations (2.15) and (2.15a) show that the phase speed of these gravity waves increases with increasing S_Z and decreases with increasing wavenumbers. The largest relative phase velocity that can be attained by these waves is that for $k = 0$, $nd_1 = \pi$ and is equal to

$$C - \bar{U} = 2d_1[gS_Z/(4\pi^2 + \beta^2 d_1^2)]^{\frac{1}{2}}. \qquad (2.15b)$$

For $d_1 = 10^4$ m, $gS_Z = 10^{-4}$ sec^{-2}, this relative phase speed is about 30 m sec^{-1}. It is seen that the effect of the mean density stratification is to reduce this relative phase speed, but this effect is very small for the earth's atmosphere.

The group velocities of these internal gravity waves are given by

$$C_{gx} = \bar{U} \pm \frac{k_1(n^2 + \frac{1}{4}\beta^2)(gS_Z)^{\frac{1}{2}}}{k(k^2 + n^2 + \frac{1}{4}\beta^2)^{\frac{3}{2}}},$$
$$C_{gy} = \pm\frac{k_2(n^2 + \frac{1}{4}\beta^2)(gS_Z)^{\frac{1}{2}}}{k(k^2 + n^2 + \frac{1}{4}\beta^2)^{\frac{3}{2}}}, \qquad (2.15c)$$
$$C_{gz} = \mp\frac{kn(gS_Z)^{\frac{1}{2}}}{(k^2 + n^2 + \frac{1}{4}\beta^2)^{\frac{3}{2}}},$$

which are always smaller than the corresponding phase velocities. It is seen that these waves are highly dispersive. Of particular interest is the fact that in the vertical, the group velocity C_{gz} and the relative phase velocity $C_z - k_1\bar{U}/n$ are in opposite directions.

From continuity consideration one expects that similar stable internal gravitational waves also exist when the mean wind has a slight vertical shear and the stratification is stable. In particular, we notice that for perturbations with $k_1 = 0$, the mean current exerts no effect, whatever the shear may be.

On the other hand, when S_Z is negative, the second part of C is purely imaginary. The amplitudes of these waves either increase or decrease exponentially with time, while all their speeds of propagation are equal to $\bar{U}$. That is to say, these growing perturbations created by unstable stratification are anchored with the mean wind $\bar{U}$. Notice further, that for a nondissipative motion the growth rate is given by k_1C_i, which is greater than zero for all wavenumbers and the maximum growth rate belongs to the perturbation of the smallest horizontal scale ($k^2 = \infty$) and is equal to $(-gS_Z)^{\frac{1}{2}}$. Thus, under such conditions, the inviscid theory is not able to define a true scale of the preferred motion.

When the dissipative terms are taken into consideration, we find from the definitions of p_1 and C in (2.12) and (2.11) that the growth rate p_r is given by

$$p_r = \frac{\pi^2\nu}{d^2}\left[\left(\frac{R\lambda^2}{1 + \lambda^2}\right)^{\frac{1}{2}} - \frac{1 + \lambda^2}{\gamma^2}\right], \qquad (2.16)$$

where $R = -gS_Z\, d^4\nu^{-2}\pi^{-4}$ is the Rayleigh number,[13] $\lambda = k\, d_1/\pi$, and γ is the ratio between the effective depth d_1 and the actual depth d. From this equation we find that a perturbation grows only when R is above a critical value given by

$$R_{0\lambda} = (1 + \lambda^2)^3/\gamma^4\lambda^2. \qquad (2.17)$$

From this we find the minimum critical R occurs at $\lambda = \frac{1}{2}$ and its value is

$$R_C = 27/4\gamma^4. \qquad (2.17a)$$

We mention that because of the requirement of the vanishing of dW/dz at a rigid boundary in addition to the vanishing of W itself, the effective depth d_1 must be smaller than the actual depth d when rigid boundaries are present. It is therefore reasonable to assign different values to the ratio $\gamma \equiv d_1/d$ for different boundary conditions. Thus, by adopting $\gamma = 1.0$ for the case of two free boundaries, $\gamma = 0.88$ for the case of one free and one rigid boundary and $\gamma = 0.788$ for the case of two rigid boundaries, Eq. (2.17a) then yields the correct critical Rayleigh numbers for these cases. The critical horizontal wavenumbers obtained from $\lambda_C' = \lambda_C\gamma^{-1}$ are also nearly correct, which shows that this simple way of including the dissipative forces works reasonably well and therefore it also can be expected to yield good results when applied to the cases where a shearing mean flow is present.

Another point worth mentioning is that when R is above R_C, a band of perturbations centered about $\lambda_0 = \frac{1}{2}$ tends to grow. The perturbation of maximum growth rate is the one with $\lambda = \lambda_0$ when $R - R_C$ is small, but shifts to a larger λ for larger $R - R_C$.

III. SUFFICIENT CONDITIONS FOR STABILITY AND INSTABILITY

In this section we apply a summary integral method introduced by Howard[14] to deduce a suffi-

[13] Since it has been assumed that the thermal conductivity is equal to the kinematic viscosity, we have the factor ν^{-2}, instead of $(\kappa\nu)^{-1}$ in R. For convenience, a factor π^{-4} is also included.

[14] L. N. Howard, J. Fluid Mech. **10**, 509 (1961).

cient condition for stability and another condition for unstable perturbations in Couette flow, which may be looked upon as an introduction to the more detailed analysis carried out in the following sections.

For our stability problem we may assume a normal mode perturbation governed by Eq. (2.12). Since the coefficients of this equation are real, we conclude immediately that if $W = W_r + iW_i$ is a solution corresponding to a complex C, $C = C_r + iC_i$, then the complex conjugate $W^* = W_r - iW_i$ is another solution corresponding to $C^* = C_r - iC_i$. Therefore, it is not necessary to differentiate between a positive and negative C_i. Assuming $C_i \neq 0$, we may use the substitution $W = V^m H$, with an unspecified m, in Eq. (2.12), where $V \equiv U - C$. We then obtain

$$(V^{2m}H_z)_z - \{(k^2 + \tfrac{1}{4}\beta^2)V^{2m} - [m(m-1)U_z^2 + gS_Z]V^{2m-2} - [(m-1)U_{zz} - \beta U_z]V^{2m-1}\}H = 0. \quad (3.1)$$

The corresponding boundary conditions are the vanishing of H at $z = z_1$ and $z = z_2$.

Multiply this equation by the complex conjugate H^* of H, integrate over the range of z and make use of the conditions that both H and H^* vanish at the boundaries, we obtain

$$\int V^{2m}Q^2 - \int \{[m(m-1)U_z^2 + g\bar{S}_Z]V^{2m-2} + [(m-1)U_{zz} - \beta U_z]V^{2m-1}\}|H|^2 = 0, \quad (3.2)$$

where $Q^2 = |H_z|^2 + (k^2 + \frac{1}{4}\beta^2)\,|H|^2$ and the integral symbol denotes integration with respect to z over the interval (z_1, z_2).

Since H is complex, the real and the imaginary parts of this equation must vanish separately; therefore it yields two integral relations for each m. In particular, the results corresponding to $m = \frac{1}{2}$, 2, and 0 are of particular interest. Thus for $m = \frac{1}{2}$ we have

$$\int (U - C_r)\left[Q^2 + (\tfrac{1}{4}U_z^2 - g\bar{S}_Z)\frac{|H|^2}{|U - C|^2}\right] + \int (\tfrac{1}{2}U_{zz} + \beta U_z)\,|H|^2 = 0, \quad (3.3a)$$

$$C_i \int \left[Q^2 - (\tfrac{1}{4}U_z^2 - g\bar{S}_Z)\frac{|H|^2}{|U - C|^2}\right] = 0. \quad (3.3b)$$

Equation (3.3b) shows that a necessary condition for the existence of amplifying waves is that the modified Richardson number $\bar{J} = (1 + k_2^2k_1^{-2})gS_ZU_z^{-2}$ must be less than $\frac{1}{4}$ in some part of the region; or, as Miles[15] expressed it, a sufficient condition for stability ($C_i = 0$) is that $\bar{J}$ is greater than $\frac{1}{4}$ everywhere, whatever the form of the profile. This result indicates that a stable stratification enhances the stability of the shearing flow, as is expected, and that three-dimensional perturbation is more stable than two-dimensional perturbation under such conditions, because it yields a larger $\bar{J}$.

Further, since $|U - C|^2 > C_i^2$, we obtain from (3.3b) that

$$k_1^2C_i^2 < \tfrac{1}{4}U_z^2 - g\bar{S}_Z, \quad (3.3c)$$

which gives an upper limit of C_i^2. However, it must not be construed from this inequality that C_i^2 will be positive when the right side is positive. This result has been obtained by Howard[14] by the integral method; it can also be derived from energy consideration.[16]

For $m = 2$, we obtain from (3.2) the two following equations:

$$\int (V_r^4 - 6V_r^2C_i^2 + C_i^4)Q^2 - \int (g\bar{S}_Z + 2U_z^2)(V_r^2 - C_i^2)\,|H|^2 = \int \frac{(\rho_0 U_z)_z}{\rho_0}(V_r^3 - 3V_rC_i^2)\,|H|^2, \quad (3.4a)$$

$$C_i\left\{\int 2V_r(V_r^2 - C_i^2)Q^2 - \int (g\bar{S}_Z + 2U_z^2)V_r\,|H|^2\right\} = C_i\int \frac{(\rho_0 U_z)_z}{2\rho_0}(3V_r^2 - C_i^2)\,|H|^2. \quad (3.4b)$$

When the Boussinesq approximation is assumed and when the mean vertical shear is constant, the terms on the right-hand side of these equations disappear. Under such conditions we can then conclude from (3.4a) that no neutral wave can exist if

$$g\bar{S}_Z < -2U_z^2, \quad (3.5)$$

because then both of the two terms on the left-hand side of (3.4a) will be positive and therefore it cannot be satisfied by any nontrivial solution. On the other hand, both of these two equations can be satisfied by certain nonzero C_i, showing that perturbations of all wavelengths will grow in a layer of Couette flow when the unstable stratification

[15] J. W. Miles, J. Fluid Mech. **10**, 496 (1961).
[16] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, Oxford, England, 1961), pp. 491–494.

is below the limiting value $g\bar{S}_Z \leq -2U_z^2$, according to the Boussinesq approximation. When the stratification is above this limit, neutral perturbations may exist.

Finally, for $m = 0$ we obtain the following two equations:

$$\int Q^2 - \int \left\{ \frac{g\bar{S}_Z}{|U - C|^4} [(U - C_r)^2 - C_i^2] - \frac{\rho_0(\rho_0^{-1}U_z)_z(U - C_r)}{|U - C|^2} \right\} |H|^2 = 0, \quad (3.6a)$$

$$C_i \int \left[\frac{g\bar{S}_z(U - C_r)}{|U - C|^4} - \frac{\rho_0(\rho_0^{-1}U_z)_z}{|U - C|^2} \right] |H|^2 = 0. \quad (3.6b)$$

Equation (3.6a) shows that, when $(\rho_0^{-1}U_z)_z = 0$, the perturbation cannot be neutral if the stratification is unstable, whereas (3.6b) shows that C_r must be equal to U at some level for unstable perturbations under the same condition.

On the other hand, for an isentropic atmosphere ($S_Z = 0$), Eq. (3.6b) shows that the necessary condition for instability is the change of sign of the expression

$$(\rho_0^{-1}U_z)_z \equiv \rho_0^{-1}(U_{zz} + \beta U_z),$$

which shows the stabilizing effect of the vertical attenuation of density.

Thus we have arrived at the results that if $gS_Z > \frac{1}{4}U_z^2$ everywhere, there can only be stable (neutral) disturbances, whereas if $g\bar{S}_Z < -2U_z^2$ and $(\rho_0 U_z)_z = 0$, there can only be unstable disturbances. The former is a sufficient condition for stability for general shearing flow while the latter is a new sufficient condition for instability of a constant shear flow. It is also necessary for a layer of infinite depth, as shown in a later section.

Judging from the stabilizing effect of the stable stratification, one expects that the continuous profiles without an inflection point will be stable for inviscid perturbations. However, no definite information concerning the influence of U_{zz} on the stability in stratified fluid has been obtained from these integral relations except for the case of an isentropic atmosphere. It seems that such information can only be obtained by investigating the solutions of the perturbation equation (2.12) for specific profiles. However, in this paper, we limit ourselves to the study of the stability of a flow of constant shear in the whole layer of a stratified fluid, and leave the other cases of shear flow for further investigation.

IV. SOLUTIONS FOR THE CASE OF MEAN FLOW OF CONSTANT SHEAR ACCORDING TO BOUSSINESQ APPROXIMATION

When the vertical shear is constant, it is convenient to use a new independent variable defined by

$$\xi = k_1(U - C)/U_z = k_1[z + (U_0 - C)/U_z]$$

where U_0 is the mean current at $z = 0$. Equation (2.12) then transforms to

$$\frac{d^2W}{d\xi^2} - \left[1 + \frac{\beta}{(k^2 + \frac{1}{4}\beta^2)^{\frac{1}{2}}} - \frac{\bar{J}}{\xi^2} \right] W = 0, \quad (4.1)$$

where $\bar{J} \equiv (1 + k_2^2 k_1^{-2})J$, $J \equiv gS_Z/U_z^2$ being the Richardson number. This equation reduces to the standard Whittaker equation by the transformation $\zeta = 2\xi$.

For most cases of interest, the term involving β in this equation may be neglected in comparison with the other terms. This is more so if we assume a finite cross-wind wavelength L_y smaller than 10 km; k_2 will then be greater than 6×10^{-4} m^{-1}. On the other hand, the value of β for earth's atmosphere is about 10^{-4} m^{-1}. Restricting our attention to such perturbations, β/k is then much less than 1 and can therefore be neglected. Equation (4.1) then reduces to

$$d^2W/d\xi^2 - (1 - \bar{J}/\xi^2)W = 0, \quad (4.1a)$$

which is the equation that would be obtained under the Boussinesq approximation, except the modification that the buoyancy effect is represented by gS_Z instead of $g\beta$. We shall therefore call (4.1a) the Boussinesq approximation of (4.1).

For a general constant $\bar{J}$, the solution of this equation is given by

$$W = \xi^{\frac{1}{2}}[AI_n(\xi) + BK_n(\xi)], \quad (4.2)$$

where $I_n(\xi)$ and $K_n(\xi)$ are Bessel functions with imaginary argument and the index n is defined by

$$n^2 = \tfrac{1}{4} - \bar{J}. \quad (4.3)$$

Application of the boundary conditions $W = 0$ at $\xi = \xi_1$ and $\xi = \xi_2$ results in two homogeneous equations for A and B. The condition for the existence of nontrivial solutions is the vanishing of the determinant of the coefficients of these two equations, which may be written as

$$C(\xi_1, \xi_2) \equiv I_n(\xi_1)K_n(\xi_2) - I_n(\xi_2)K_n(\xi_1) = 0. \quad (4.4)$$

From the roots of this characteristic equation we find C_r and C_i as functions of the wavenumber k and $\bar{J}$.

In the following sections we at first discuss very briefly the stability of the perturbations for the case of an infinite layer bounded below, and then discuss in more detail the character of the perturbations in a layer of finite depth bounded both above and below by horizontal surfaces.

V. PERTURBATION IN AN INFINITE LAYER

If the fluid layer is bounded below by a horizontal surface but is of infinite depth, we must set A to zero in (4.2) since $I_n(\xi)$ behaves as e^{ξ} for large ξ. Thus for this case the boundary requirement that W vanishes at $z = 0$ is satisfied if $\xi_0 = k(U_0 - C)/U_z$ is a root of

$$K_n(\xi_0) = 0. \tag{5.1}$$

It is known[17] that $K_n(\xi)$ has no zeros for which $|\arg \xi| \leq \frac{1}{2}\pi$, and that it has no negative real zeros unless $n = 2j + \frac{3}{2}$, where j is a positive integer, including zero. Thus when $n - \frac{1}{2}$ is not equal to an odd integer, all the zeros of $K_n(\xi)$ are complex with a negative real part. It is also known that the total number N_K of zeros of $K_n(\xi)$ is equal to the even integer nearest to $n - \frac{1}{2}$. When $n - \frac{1}{2}$ is an integer, N_k is exactly equal to $n - \frac{1}{2}$. The number N_k changes only when n crosses one of the values $2j + \frac{3}{2}$. As n increases from $2j + \frac{3}{2} - \epsilon$ to $2j + \frac{3}{2}$, a negative real zero appears and this zero splits up into a pair of conjugate complex zeros when n increases from $2j + \frac{3}{2}$ to $2j + \frac{3}{2} + \epsilon$. Thus for $n < \frac{3}{2}$, $K_n(\xi)$ has no zeros, and when $n = \frac{3}{2}$, it has one real zero given by $\xi_0 = -1$, while for $\frac{3}{2} < n < \frac{7}{2}$, it has 2 complex zeros.

Since a complex ξ_0 implies a complex C, the stability criterion for a shearing layer of infinite depth is $n > \frac{3}{2}$. In terms of the modified Richardson number $\bar{J}$, this criterion is given by

$$\begin{aligned} \bar{J} &> -2 \quad \text{stable}, \\ \bar{J} &< -2 \quad \text{unstable}. \end{aligned} \tag{5.2}$$

This criterion confirms the conclusion (3.5) obtained from the integral relation that the perturbation cannot be neutral when $\bar{J}$ is less than -2.

Note that this criterion is independent of k_1 for two-dimensional perturbations ($k_2 = 0$), therefore disturbances of all scales will become unstable simultaneously according to this inviscid approximation, if they are really independent of y.

Since neutral solution exists for real n only when n is a half integer, no eigensolution exists for $0 < n < \frac{3}{2}$, which corresponds to $0.25 > \bar{J} > -2$. According to the results obtained by Case,[6] which can be carried over directly to the present case for large t, that, for this range of n, small initial temperature perturbations will grow or decay at the rate of $t^{(n-\frac{1}{2})}$, while initial vorticity perturbations will behave like $t^{(n-\frac{3}{2})}$ for large t. Therefore they will be damped out, even though slowly, for inviscid motion.

When $\bar{J}$ is greater than $\frac{1}{4}$, that is, for strong gravitational stability and (or) weak shear, $n = i\mu$ is purely imaginary. The function $K_{i\mu}(\xi)$ has infinity simple positive real zeros but no negative real zero nor any complex zero. Thus, for this case, we have a discrete infinite set of eigenvalues, and the corresponding eigensolutions represent the stable gravity waves in an infinite layer of constant shear flow. All these waves propagate backward relative to the mean current at the level $z = 0$, and the longest wave has the highest backward velocity.

VI. ELEMENTARY SOLUTIONS FOR A LAYER OF FINITE DEPTH

For a layer of finite depth it is more convenient to shift the origin of z to midway between the two boundaries. We then have

$$\xi_1 = -\alpha(1 + q), \qquad \xi_2 = \alpha(1 - q), \tag{6.1}$$

where $\alpha = h(k_1^2 + k_2^2)^{\frac{1}{2}}$ is the dimensionless vectorial horizontal wavenumber, $q = (C - U_0)(hU_z)^{-1}$ is the dimensionless phase velocity relative to the midlevel mean velocity U_0, and h is the half-depth of the layer.

For this case it is rather difficult to obtain the zeros of $C(\xi_1, \xi_2)$ of (4.4) directly, except when n is a half-integer, where the solutions $I_n(\xi)$ and $K_n(\xi)$ reduce to elementary functions. We shall, therefore, first investigate the eigenvalues and the eigensolutions for these particular values of n. Since the zero λ of $C(\xi_1, \xi_2)$ is an analytic function of α and n, we can use these results as a guide to the approximate results we are going to obtain in the following sections for other values of n.

Setting $n = m - \frac{1}{2}$ in Eq. (4.1a) it then becomes

$$\frac{d^2W}{d\xi^2} - \left[1 + \frac{m(m-1)}{\xi^2}\right]W = 0. \tag{6.2}$$

The general solution of this equation for an integer m is given by

$$W = \xi^m\left(\frac{1}{\xi}\frac{d}{d\xi}\right)^m (C_1e^{\xi} + C_2e^{-\xi}), \tag{6.3}$$

[17] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, England, 1948), pp. 511–513.

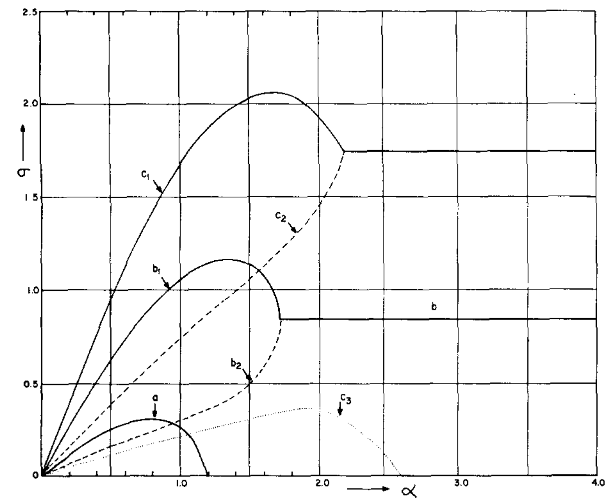

FIG. 1. Variation of the amplification factor σ as a function of the vectorial horizontal wavenumber α. Curve a for $n = 1.5$; curves b_1, b_2 for $n = 2.5$; curves c_1, c_2, c_3 for $n = 3.5$.

and, therefore, it involves two polynomials of degree $(m - 1)$ in ξ^{-1}. The solutions for $m = 1, 2, 3, 4$ suffice to illustrate the general nature of the perturbations.

(a) $m = 1$, corresponding to $n = 0.5$, $\bar{J} = 0$, or constant shearing flow in a neutral fluid. The solution for this case is given by

$$W = C_1 e^{\xi} + C_2 e^{-\xi}. \tag{6.4}$$

The two boundary conditions cannot be satisfied simultaneously by this solution, showing that the time variation of the perturbations cannot be represented by an exponential function of t. It will be shown later that this case is stable for all perturbations.

(b) $m = 2$, corresponding to $n = 1.5$, $\bar{J} = -2$. The general solution is

$$W = C_1 e^{\xi}\left(1 - \frac{1}{\xi}\right) + C_2 e^{-\xi}\left(1 + \frac{1}{\xi}\right). \tag{6.5}$$

Applying the boundary conditions at $\xi = \xi_1$ and $\xi = \xi_2$ and eliminating C_1 and C_2 we obtain the following characteristic equation:

$$\alpha^2 q^2 = \alpha^2 + 1 - 2\alpha \coth 2\alpha, \tag{6.6}$$

which shows that q is purely imaginary for $0 < \alpha < 1.1997$ and is purely real for $\alpha > 1.1997$. Thus for $n = 1.5$, all the perturbations with $\alpha < 1.1997$ are exponentially amplifying, while those with $\alpha > 1.1997$ represent neutral waves propagating relative to the midlevel mean current.

The dimensionless amplification factor $\sigma(\equiv \alpha q_i)$ of the amplifying perturbations $(0 < \alpha < 1.1997)$ is illustrated in Fig. 1 by curve a. It is seen that σ has a maximum value of 0.3100 at $\alpha = 0.8031$, after which it decreases to zero for $\alpha \geq 1.1997$.

From symmetry we know that if there exists a disturbance which moves forward relative to the midlevel current, there must also exist another disturbance which moves backward with the same speed. We need, therefore, consider only the absolute value of q_r.

From (6.6) we see that for large α such that $\coth 2\alpha \doteq 1$ we have

$$q = q_r = \pm(1 - 1/\alpha). \tag{6.6a}$$

Therefore, the asymptotic value of $|q_r|$ for very large α is 1, but $|q_r|$ is less than 1 for all finite α. The variation of $|q_r|$ with α is represented in Fig. 2 by curve a. It is seen that it increases at first very rapidly from the point $\alpha = 1.1997$ and then very slowly toward its asymptotic value 1.

(c) $m = 3$, which corresponds to $n = 2.5$, $\bar{J} = -6$. The general solution for this case is

$$W = C_1 e^{\xi}\left(1 - \frac{3}{\xi} + \frac{3}{\xi^2}\right) + C_2 e^{-\xi}\left(1 + \frac{3}{\xi} + \frac{3}{\xi^2}\right), \tag{6.7}$$

while the characteristic equation is

$$y^2 + 3by + 12\alpha^2 - 9b = 0, \tag{6.8}$$

with $y = \alpha^2(q^2 - 1)$, $b = 2\alpha \coth 2\alpha - 1$.

Equation (6.8) has two distinct real roots in y for $0 < \alpha < 1.725$, and it has two conjugate complex roots for $\alpha > 1.725$. Both of the two real roots yield purely imaginary q, but one gives a much higher amplification factor than the other, as is seen from the curves b_1 and b_2 in Fig. 1. Further,

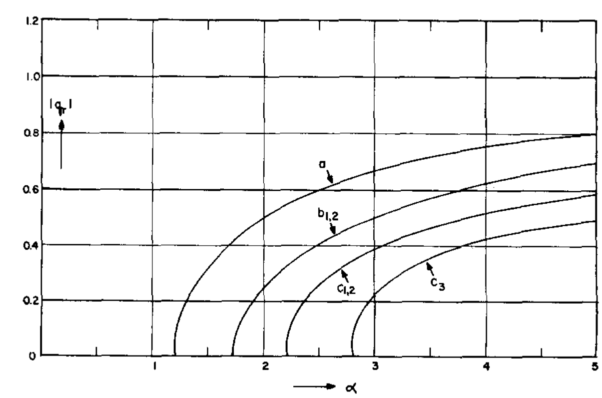

FIG. 2. Variation of the absolute value of the nondimensional phase velocity as a function of α. Curve a for $n = 1.5$; curves $b_{1,2}$ for $n = 2.5$; curves $c_{1,2}$ and c_3 for $n = 3.5$.

the amplification factor σ as given by the first root has a maximum at $\alpha = 1.35$, after which it decreases with increasing α until it joins the curve b_2, which is given by the second root. The amplification rate represented by b_2 increases monotonicly with increasing α, at first very slowly, and then rapidly, until it joins b_1.

On the other hand, the two (conjugate) complex roots for $\alpha > 1.725$ yield complex q with the same real part and the same imaginary part, and, therefore, may be considered as being identical, even though they still make a total of four different combinations by taking $\pm q_r$ and $\pm q_i$.

One oddity of these short transitive disturbances is that their growth rate σ is almost independent of α. For the present case, the σ for these shorter perturbations is nearly equal to its asymptotic value 0.866, as is seen from curve b in Fig. 1. This asymptotic amplication rate can be obtained from (6.8) by setting $\coth 2\alpha = 1$.

The phase velocity of these short amplifying waves is illustrated in Fig. 2 by curve b. The trend of variation is similar to that for $m = 2$ except that it starts from a larger α. The asymptotic relation is given by

$$q_r = \pm(1 - 3/2\alpha), \tag{6.8a}$$

and therefore it also reaches ± 1 for very large α.

(d) $m = 4$, which corresponds to $n = 3.5$, $\bar{J} = -12$. The general solution for this case is given by

$$W = C_1 e^{\xi}\left(1 - \frac{6}{\xi} + \frac{15}{\xi^2} - \frac{15}{\xi^3}\right) + C_2 e^{-\xi}\left(1 + \frac{6}{\xi} + \frac{15}{\xi^2} + \frac{15}{\xi^3}\right), \tag{6.9}$$

while the corresponding characteristic equation is

$$y^3 + 6by^2 + (60\alpha^2 - 45b)y + (225 + 60\alpha^2)b - 300\alpha^2 = 0, \tag{6.10}$$

where y and b are as defined in (6.8).

For $0 < \alpha < 2.2$, all the three roots of (6.10) are real and they all give pure imaginary q, therefore these perturbations are amplifying and stationary with respect to the mean current U_0.

The amplification rates σ given by these three solutions are illustrated in Fig. 1 by the curves c_1, c_2, and c_3, respectively. It is seen that the σ given by the first root is much higher than that given by the other two roots, and that this σ has a maximum at $\alpha = 1.70$. After this point σ decreases with increasing α until it joins the curve c_2 at $\alpha = 2.2$. For $\alpha > 2.2$, these two roots become a conjugate complex pair, and the perturbations represented by them become transitive and their amplification rates become nearly independent of α, as for the case $m = 3$.

The σ values given by the third root of (6.10) is represented by curve c_3 in Fig. 1. It is much smaller than that given by the other two roots. This curve has its maximum at $\alpha = 2.0$ and it becomes zero for $\alpha > 2.56$.

The phase velocity of the short transitive perturbations is given by curves $c_{1,2}$ and c_3 in Fig. 2. For very large α, $|q_r|$ also approaches 1 from below.

VII. POWER SERIES AND ASYMPTOTIC APPROXIMATIONS OF THE SOLUTION OF (4.1a)

For n not equal to a half-integer, we use the ascending power-series expansions of the Bessel functions for relatively long waves (small α) and asymptotic expansions for short waves (large α). By taking enough terms in these expansions, the perturbations of intermediate wavelengths can also be represented. When used in conjunction with the results obtained in the preceding section for half-integer n, we obtain a complete picture of the nature of the perturbations.

A. Power-Series Approximation for Relatively Long Waves

Assuming n is not an integer, we may use the function $I_{-n}(\xi)$ as the second solution instead of $K_n(\xi)$. We then have

$$\begin{aligned} I_n(\xi) &= \frac{1}{n!}\left(\frac{\xi}{2}\right)^n\left[1 + \frac{1}{(n+1)}\frac{\xi^2}{2^2} + \frac{\xi^4}{2^4\cdot 2!(n+1)(n+2)} + \cdots\right], \\ I_{-n}(\xi) &= \frac{1}{(-n)!}\left(\frac{\xi}{2}\right)^n\left[1 - \frac{1}{(n-1)}\frac{\xi^2}{2^2} + \frac{\xi^4}{2^4\cdot 2!(n-1)(n-2)} - \cdots\right]. \end{aligned} \tag{7.1}$$

Substituting these functions in the characteristic equation (4.4) with $I_{-n}(\xi)$ in place of $K_n(\xi)$, we then obtain

$$(\xi_2^{2n}-\xi_1^{2n})\left[1-\frac{\xi_1^2\xi_2^2}{16(n^2-1)}+\frac{\xi_1^4\xi_2^4}{2^{10}(n^2-1)(n^2-4)}\right]+\frac{(\xi_2^{2n+2}-\xi_1^{2n+2})}{4(n+1)}\cdot\left[1-\frac{\xi_1^2\xi_2^2}{32(n-1)(n+2)}\right]-\frac{\xi_1^2\xi_2^2}{4(n-1)}(\xi_2^{2n-2}-\xi_1^{2n-2})\left[1-\frac{\xi_1^2\xi_2^2}{32(n+1)(n-2)}\right]$$
$$+\frac{\xi_2^{2n+4}-\xi_1^{2n+4}}{32(n+1)(n+2)}\left[1-\frac{\xi_1^2\xi_2^2}{48(n-1)(n+3)}\right]+\frac{(\xi_1^4\xi_2^{2n}-\xi_2^4\xi_1^{2n})}{32(n-1)(n-2)}\cdot\left[1-\frac{\xi_1^2\xi_2^2}{48(n+1)(n-3)}\right]-\frac{\xi_1^6\xi_2^{2n}-\xi_2^6\xi_1^{2n}}{384(n-1)(n-2)(n-3)}+\frac{\xi_2^{2n+6}-\xi_1^{2n+6}}{384(n+1)(n+2)(n+3)}$$
$$+\frac{\xi_2^{2n+8}-\xi_1^{2n+8}}{2^8\cdot 4!(n+1)\cdots(n+4)}+\frac{\xi_1^8\xi_2^{2n}-\xi_2^8\xi_1^{2n}}{2^8\cdot 4!(n-1)\cdots(n-4)}+\cdots=0. \tag{7.2}$$

For very small α, all the higher-order terms may be neglected. This equation then reduces to

$$\xi_2^{2n}-\xi_1^{2n}=0. \tag{7.3}$$

Putting $q=q_r+iq_i$, $\xi_j=r_je^{i\theta_j}$, $j=1,2$, (7.3) then becomes

$$e^{2n(\theta_2-\theta_1)i}=(r_1/r_2)^{2n}.$$

Since r_1/r_2 is real and positive, we must have

$$2n(\theta_2-\theta_1)=2m\pi, \tag{7.4}$$

where m is a positive integer. This result implies $r_1=r_2$, and therefore $q_r=0$, which means that the very long waves move with the wind at the midlevel.

Without losing generality, we may assume $q_i>0$, we then have $-\frac{1}{2}\pi\le\theta_2\le 0$, $-\pi\le\theta_1\le-\frac{1}{2}\pi$ and therefore $\theta_2-\theta_1\le\pi$. Thus in order that (7.4) can be satisfied, we must have

$$n\ge 1. \tag{7.5}$$

This agrees with Hoiland's result for the singular perturbation $k_1=0$.

To discuss the stability of the perturbations of finite wavelength, we must retain the higher-order terms and solve for q_r and q_i as functions of α and n. However, as has already been demonstrated by the solutions for half-integer n, and it can also be shown by the α^{2n+2}-order and the α^{2n+4}-order approximations of (7.2), that the fast-growing long waves move with the midlevel mean current. Thus, equating the real and the imaginary parts of the α^{2n+2}-order approximation of (7.2) to zero separately, we obtain the two following relations:

$$\frac{n^2-1}{\alpha^2}-\tfrac{1}{2}(1+q_r^2+q_i^2)-\frac{q_r(R^2-1)-2Rq_i\sin 2n(\pi-\mu)}{|R-e^{2n(\pi-\mu)i}|^2}n=0, \tag{7.5a}$$

$$q_rq_i+\frac{q_i(R^2-1)+2Rq_r\sin 2n(\pi-\mu)}{|R-e^{2n(\pi-\mu)i}|^2}n=0, \tag{7.5b}$$

where $R=\rho^{2n}$ and ρ and μ are defined by

$$(1-q)/(1+q)=\rho e^{i\mu}.$$

Eliminating $\sin 2n(\pi-\mu)$ from these two relations we obtain

$$q_r\left[\frac{n^2-1}{\alpha^2}-\tfrac{1}{2}(1+q_r^2+q_i^2)-\frac{n(R^2-1)(q_r^2+q_i^2)}{q_r\,|R-e^{2n(\pi-\mu)i}|^2}\right]=0, \tag{7.6}$$

which shows that $q_r=0(R=1)$ is the root of this equation.

Taking this result as being valid in general for the relatively long waves, Eq. (7.2) then becomes an equation in q_i alone. Since q_i is assumed to be positive and $q_r=0$, we have $r_1^2=r_2^2=\alpha^2(1+q_i^2)\equiv r^2$, $q_i=-\tan\theta_2$, $\theta_1+\theta_2=-\pi$. Setting $\beta=\pi+2\theta_2$ and omitting a common factor $r^{2n}e^{-n\pi i}$, Eq. (7.2) then reduces to

$$\sin n\beta-\frac{r^2}{4}\left[\frac{\sin(n+1)\beta}{n+1}-\frac{\sin(n-1)\beta}{n-1}\right]+\frac{r^4}{32}\left[\frac{\sin(n+2)\beta}{(n+1)(n+2)}+\frac{\sin(n-2)\beta}{(n-1)(n-2)}-\frac{2\sin n\beta}{n^2-1}\right]+\frac{r^6}{128(n^2-1)}\left[\frac{\sin(n+1)\beta}{n+2}-\frac{\sin(n-1)\beta}{n-2}\right]$$
$$-\frac{r^6}{384}\left[\frac{\sin(n+3)\beta}{(n+1)(n+2)(n+3)}-\frac{\sin(n-3)\beta}{(n-1)(n-2)(n-3)}\right]$$
$$+\frac{r^8}{1536}\left\{\frac{1}{(n+1)(n+2)(n+3)}\left[\frac{\sin(n+2)\beta}{n-1}-\frac{\sin(n+4)\beta}{4(n+4)}\right]+\frac{1}{(n-1)(n-2)(n-3)}\cdot\left[\frac{\sin(n-2)\beta}{n+1}-\frac{\sin(n-4)\beta}{4(n-4)}\right]\right\}+\cdots=0. \tag{7.7}$$

This equation converges rapidly for small α, so that all terms higher than r^8 can be neglected for $\alpha < 1.2$. Values of the amplification factor σ have been computed from this equation and they are illustrated in Fig. 3, and the method of computations is described below.

At first we use only the first two terms of Eq. (7.7) to determine a first approximation of αq_i for given values of α and n, and then obtain higher approximations by including more and more higher-order terms. From the first two terms we notice at once that for $n > 1$, $n\beta$ must be greater than π, and that for n slightly greater than 1 and for small α, $n\beta$ is very close to π. In fact, one may just put $n\beta = \pi$ and obtain a zero-order estimate of θ_2 and then obtain q_i from $\tan \theta_2 = -q_i$. It is then evident that the first approximation requires $n\beta$ to be slightly greater than π and this value can easily be found. Using this approximation in the second-order terms, one can easily estimate the correction that is required for a second-order approximation. The procedure is repeated until the whole Eq. (7.7) is satisfied to the desired degree of accuracy.

B. Short Waves

For large α, the fundamental solutions $I_n(\xi)$ and $K_n(\xi)$ may be represented by their asymptotic approximations:

$$\begin{aligned} I_n(\xi) &= \frac{e^{\xi}}{(2\pi\xi)^{\frac{1}{2}}}\left[1 + \frac{1-4n^2}{1!8\xi} + \frac{(1-4n^2)(9-4n^2)}{2!8^2\xi^2} + \cdots\right], \\ K_n(\xi) &= \frac{e^{-\xi}}{(2\pi\xi)^{\frac{1}{2}}}\left[1 - \frac{1-4n^2}{1!8\xi} + \frac{(1-4n^2)(9-4n^2)}{2!8^2\xi^2} - \cdots\right]. \end{aligned} \tag{7.8}$$

Substituting in (4.4) we find that $C(\xi_1, \xi_2)$ is given by

$$C(\xi_1, \xi_2) = -\frac{1}{\pi(\xi_1\xi_2)^{\frac{1}{2}}} \cdot \left[\sinh 2\alpha - \frac{1-4n^2}{4\xi_1\xi_2}\alpha \cosh 2\alpha + \cdots\right]. \tag{7.9}$$

Now if $0 \leq |q_r| < 1$, such that $\xi_{1r} \to -\infty$ and $\xi_{2r} \to +\infty$ as $\alpha \to \infty$, $C(\xi_1, \xi_2)$ will become infinitely large and the boundary conditions can therefore not be satisfied. On the other hand, if $|q_r| \to 1$ as $\alpha \to \infty$, it can then be shown that the boundary conditions will be satisfied when $n \geq \frac{3}{2}$.

To prove this statement, let us at first assume $q_r \to -1$ through $-1 + \epsilon$. Then the real part of ξ_2 approaches ∞ as $\alpha \to \infty$ while ξ_1 remains finite and has a negative real part. We may therefore write the characteristic equation (4.4) in the form

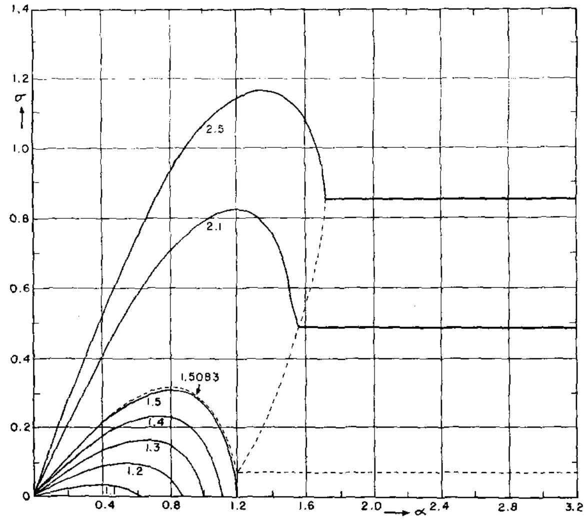

FIG. 3. The variation of the amplification factor σ as a function of α for various values of n.

$$\begin{aligned} K_n(\xi_1) &= [K_n(\xi_2)/I_n(\xi_2)]I_n(\xi_1) \\ &\simeq e^{-2\xi_2}I_n(\xi_1) \to 0 \end{aligned} \tag{7.10}$$

for large α. Thus the condition is satisfied if ξ_1 is a zero of $K_n(\xi_1)$, as for the case of a layer of infinite depth. As shown in Sec. V, such zeros exist only when n is equal to or greater than 1.5, and that all these roots have a negative real part. The same argument applies when $q_r \to 1$ through $1 - \epsilon$, as $I_n(-\xi)$ is a constant multiple of $K_n(\xi)$. Therefore, the criterion for infinitely large α is $n = 1.5$, which is the result obtained by Eliassen, Hoiland, and Riis.[9] Comparing this result with that following (6.6), where it was found that all perturbations with $\alpha > 1.1997$ are neutral for $n = 1.5$, there must either be a contradiction or all perturbations with $\alpha >$ 1.1997 must become excited at once when n crosses the value 1.5 from below, which is rather uncommon for shearing flow instability.

We shall now show that this is actually the case, that is, all the perturbations are amplified when $n = 1.5 + \epsilon$, with ϵ being a small but positive number. To prove this, let us make use of the asymptotic expressions (7.8) for $I_n(\xi)$ and $K_n(\xi)$. It may then be assumed that all terms containing ξ^{-4} and higher negative powers of ξ can be neglected in the characteristic equation for all $\alpha \geq 1$. We then find that this equation reduces to the following:

$$a_0y^4 + a_1y^3 + a_2y^2 + a_3y + a_4 = 0, \tag{7.11}$$

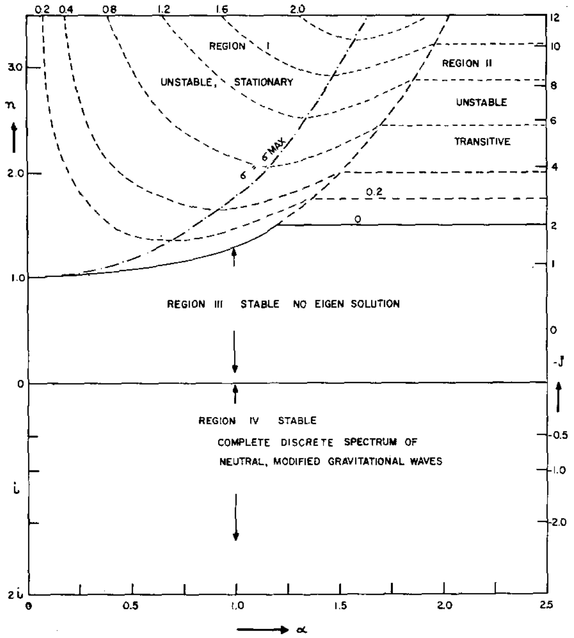

FIG. 4. The four different regimes of perturbation and the distribution of the amplification factor σ in the unstable regions I and II as functions of α and n.

with

$$y = \alpha^2(q^2 - 1), \qquad a_0 = [1 + (\epsilon/8)]^{-1},$$

$$a_1 = 2\alpha \coth 2\alpha - 1,$$

$$a_2 = \frac{\epsilon}{8}\left\{2\alpha^2 + \frac{5}{6} - \frac{9}{4}\left(1 - \frac{\epsilon}{24}\right) 2\alpha \coth 2\alpha + \frac{7\epsilon}{120} - \frac{\epsilon^2}{960}\right\}, \quad (7.11\text{a})$$

$$a_3 = -\frac{\epsilon}{8}\alpha^2\left[\frac{8}{3}\left(1 - \frac{\epsilon}{16}\right)a_1 - \frac{2\epsilon}{15} + \frac{\epsilon^2}{120}\right],$$

$$a_4 = \frac{\epsilon}{6}\alpha^4\left(1 - \frac{7\epsilon}{80} + \frac{\epsilon^2}{640}\right).$$

From (7.11a) we find that the discriminant Δ of (7.11) is negative for small positive ϵ, therefore there are always two complex and two real roots for y. The two complex roots always give complex q, therefore amplifying perturbations exist for all α when $n = \frac{3}{2} + \epsilon$.

The results corresponding to $\epsilon = 0.1$ ($n = 1.50831$) is represented in Fig. 3 by the dashed curve, which shows that all the perturbations with $\alpha > 1.1997$ are amplifying exponentially at nearly the same rate $\sigma \simeq 0.070$. The long-wave part of this curve is obtained from the power-series approximation (7.7). These two approximations agree quite well in the intermediate region $\alpha \simeq 1.0$.

The results obtained above show that when n is greater than 1.5 (i.e., for $\bar{J} < -2$), all perturbations are unstable, whereas when $1 < n < 1.5$, only the perturbations within a long-wave band can grow and that there exists a particular wavelength which grows fastest. The limiting wavenumber α_0 which marks the short-wave-side cutoff is accurately given by the α^4-order approximation of (7.2) or (7.7), viz:

$$\alpha_0^2 = \tfrac{2}{3}(4 - n^2)\{[1 + 6(n^2 - 1)/(4 - n)^2]^{\frac{1}{2}} - 1\} \quad (7.12)$$

for $1 < n < \frac{3}{2}$. Conversely, we may also solve for the limiting $n = n_0(\alpha)$ as a function of α from this equation.

This critical curve, which separates the amplifying region from the stable region, is illustrated in Fig. 4, which also depicts the variation of σ as a function of α and n or $\bar{J}$.

In this graph, four different regimes have been differentiated. Region 1 is bounded by the critical curve $n = n_0(\alpha)$ below and a sloped curve on the right which separates it from region II. This region is the region of long, amplifying stationary perturbations which behave like ordinary convection. A wavelength of maximum growth rate exists in this region. Region II is bounded below by $n = 1.5$. In this region, σ is nearly independent of the wavenumber α, and the perturbations are amplifying and transitive, therefore they behave like gravity waves. Below the critical curve is the stable region, which is composed of two regions. Region III is bounded by the critical curve above, and the horizontal line $n = 0$, $\bar{J} = 0.25$ below; it is a stable region where no nonsingular eigensolutions exist. Region IV is the region of the modified gravitational waves. The nature of the solutions in these two stable regions is examined in the two following sections.

VIII. PERTURBATION FOR STABLE STRATIFICATION AND WEAK UNSTABLE STRATIFICATION

It is shown in Sec. IV that exponentially unstable perturbations exist only in the region $n > n_0$, which starts from $n_0 = 1$ for very long waves, to $n_0 = 1.5$ for the short waves ($\alpha > 1.1997$). Below this neutral curve, the perturbations must be represented either by superposition of neutral solutions of Eq. (4.1a), or by superposition of singular solutions if no eigensolution exists. In this section we at first show that Eq. (4.1a) has no continuous eigensolutions for $0 < n < n_0$, and secondly we investigate the behavior of arbitrary initial perturbations for large t.

For $\alpha < 1.2$, it suffices to use the α^{2n+4}-order approximation of Eq. (7.2). Putting $X = (1-q)(1+q)^{-1}$ we then obtain

$$X^{2n}\left\{1+\frac{\alpha^2}{2}\left[\frac{(1-q)^2}{n+1}-\frac{(1+q)^2}{n-1}\right]+\frac{\alpha^4}{32}\left[\frac{(1-q)^4}{(n+1)(n+2)}+\frac{(1+q)^4}{(n-1)(n-2)}-\frac{2(1-q^2)^2}{n^2-1}\right]\right\}=e^{2n\pi i}\left\{1+\frac{\alpha^2}{2}\left[\frac{(1+q)^2}{n+1}-\frac{(1-q)^2}{n-1}\right]+\frac{\alpha^4}{32}\left[\frac{(1+q)^4}{(n+1)(n+2)}+\frac{(1-q)^4}{(n-1)(n-2)}-\frac{2(1-q^2)^2}{n^2-1}\right]\right\}. \quad (8.1)$$

For $n \leq n_0(\alpha)$, q must be real if eigensolutions do exist. X is then also real, and therefore the left side of this equation is real for $0 < n < n_0$. On the other hand, $e^{2n\pi i}$ is complex when n is not a half-integer; therefore the coefficients of X^{2n} and of $e^{2n\pi i}$ must vanish separately. Taking the difference and sum of these two expressions we then obtain

$$\frac{\alpha^2 q}{n^2-1}\left\{2n-\frac{\alpha^2}{16(n^2-4)}[q+6n(1+q^2)]\right\}=0, \quad (8.2a)$$

$$1-\frac{\alpha^2}{2}\frac{1+q^2}{n^2-1}+\frac{\alpha^4}{8(n^2-1)(n^2-4)}\cdot[3(1+q^4)+(4n+3)q^2]=0. \quad (8.2b)$$

These equations show that the possible solutions are characterized by $q = q_r = 0$. Equation (8.2b) then reduces to

$$1-\frac{1}{2}\frac{\alpha^2}{n^2-1}+\frac{3\alpha^4}{8(n^2-1)(n^2-4)}=0. \quad (8.2c)$$

This equation yields the critical curve only, and there is no solution for any given n other than $n = n_0(\alpha)$. Therefore, no eigensolution exists for $0 < n < n_0$.

In order to examine the behavior of the perturbations for this case, we must treat it as an initial-value problem. Following a procedure used by Case[6] in discussing the stability of shearing flow in a stably stratified fluid, we shall solve Eq. (2.11) by making use of the singular solutions of (4.1a) and then inverting the Laplace and the Fourier transforms. Thus, we have

$$w_{k_1k_2}(z,t)=\frac{\exp\left[-\nu(k^2+\pi^2/d_1^2)t\right]}{2\pi i\alpha}\cdot\int d\xi_0\int_C \exp(p_1t)\,dp_1\,G(\xi,\xi_0)\cdot\left[\frac{X_1}{\xi_0-i\lambda}+\frac{X_2}{(\xi_0-i\lambda)^2}\right], \quad (8.3)$$

where $G(\xi, \xi_0)$ is the Green's function of the problem, $\lambda = p_1k/k_1U_z$, and X_1 and X_2 denote the initial vorticity and thermal perturbations, given respectively by

$$X_1 = -(ki/k_1U_z)[(d^2W_0/dz^2)-k^2W_0], \quad (8.3a)$$

$$X_2 = (gk^2/k_1^2U_z^2)s_0.$$

For the present problem, the Green's function is given by

$$G_{1,2}(\xi,\xi_0)=\tfrac{1}{2}\pi(l_0l)^{\frac{1}{2}}[K_n(l_{1,2})I_n(l)-K_n(l)I_n(l_{1,2})]\cdot\frac{[K_n(l_0)I_n(l_{2,1})-K_n(l_{2,1})I_n(l_0)]}{K_n(l_2)I_n(l_1)-K_n(l_1)I_n(l_2)} \quad (8.4)$$

with $l = \xi - i\lambda$, $l_0 = \xi_0 - i\lambda$, $l_1 = -i\lambda$, $l_2 = \alpha - i\lambda$, and $l_{1,2}$ and $l_{2,1}$ stand either for l_1 or l_2. The first expression is to be used for $0 < \xi_0 < \xi$, that is, with $l_{2,1} = l_2$ and $l_{1,2} = l_1$, whereas for $\xi < \xi_0 < \alpha$ we use the second expression, viz., $l_{1,2} = l_2$, $l_{2,1} = l_1$.

In the integration over ξ_0 in (8.3), the largest contribution is from the region of small l_0; we may therefore replace $K_n(l_0)$ and $I_n(l_0)$ in (8.4) by the first terms of their power-series expansions and obtain

$$G_{1,2}(\xi,\xi_0)\simeq 2^{n-1}(n-1)!\,l^{\frac{1}{2}}l_0^{\frac{1}{2}-n}\cdot\frac{[K_n(l_{1,2})I_n(l)-K_n(l)I_n(l_{1,2})]}{[K_n(l_2)I_n(l_1)-K_n(l_1)I_n(l_2)]}. \quad (8.4a)$$

The integration over p_1 can now be carried out. Taking the contour C just right of the imaginary axis, and considering the thermal perturbation alone, we then have

$$\frac{1}{2\pi i}\int_C\frac{e^{p_1t}\,dp_1}{(\xi_0-i\lambda)^{n+\frac{3}{2}}}=\left(\frac{ik_1U_z}{k}\right)^{n+\frac{3}{2}}\frac{t^{n+\frac{1}{2}}}{(n+\frac{1}{2})!}e^{-ik_1U_z\xi_0t/k}. \quad (8.5)$$

Integrating this quantity with respect to ξ_0 we then find

$$w_{k_1,k_2}(z,t)\sim t^{n-\frac{1}{2}}\exp\left[-\nu(k^2+\pi^2/d_1^2)t\right]. \quad (8.6)$$

w itself also behaves like $t^{n-\frac{1}{2}}$ times the exponential viscous damping factor for large t when X_2 is well behaved at $k_1 = 0$, as in the case of an infinite depth.[6]

The analysis of the initial vorticity perturbations proceeds in the same manner. It is found that such perturbations behave like $t^{n-1.5}$ times the exponential damping factor for large t and therefore will be reduced to zero even without the dissipative effect.

IX. SOLUTIONS FOR STRONG STABLE STRATIFICATION AND WEAK SHEAR, $\bar{J} > 1/4$

In this case $n = i\mu$ is purely imaginary and the functions $I_n(\xi)$ and $I_{-n}(\xi)$ are both well defined; it is then convenient to use these two functions as the fundamental solutions.

Since the coefficient of ξ^r in the power-series expansion of $I_n(\xi)$ is complex and ξ is real, we may separate $I_n(\xi)$ into a real part and an imaginary part such that

$$I_n(\xi) = \left(\frac{\xi}{2}\right)^{i\mu} \sum_{r=0}^{\infty} \rho_r e^{i\theta_r} \xi^r = R(\xi) e^{i[N(\xi)+\epsilon\mu\pi i]} \tag{9.1}$$

with

$$R^2(\xi) = \tfrac{1}{4} \sum_r \rho_r^2 \xi^{2r} + \tfrac{1}{2} \sum_{r_1} \sum_{r_2} \rho_{r_1}\rho_{r_2} \xi^{r_1+r_2} \cos(\theta_{r_2} - \theta_{r_1}),$$
$$\tan N = \frac{\sum_r \rho_r(\frac{1}{2}\xi)^r \sin(\theta_r + \mu \log|\frac{1}{2}\xi|)}{\sum_r \rho_r(\frac{1}{2}\xi)^r \cos(\theta_r + \mu \log|\frac{1}{2}\xi|)}, \tag{9.2a}$$

where ρ_r and θ_r are given by

$$\rho_r e^{i\theta_r} = \frac{1}{r!(r+i\mu)!} \equiv C_r + id_r. \tag{9.2b}$$

In (9.1) we have $\epsilon = 0$ for positive ξ and $\epsilon = 1$ for negative ξ.

Since ξ is real, $I_{-i\mu}(\xi)$ is the conjugate complex of $I_{i\mu}(\xi)$ and therefore the general solution of Eq. (4.1a) is given by

$$W = \xi^{\frac{1}{2}} R(\xi)[Ae^{i(N+\epsilon\mu\pi i)} + Be^{-i(N+\epsilon\mu\pi i)}]. \tag{9.3}$$

Since $I_n(\xi)$ and $I_{-n}(\xi)$ are two linearly independent solutions of (4.1a), they cannot vanish simultaneously for any ξ; therefore, $R(\xi)$ has no zeros. Further, $\xi^{\frac{1}{2}}$ has no zero other than the origin. Therefore, the boundary conditions require that the function in the bracket of (9.3) vanishes at ξ_1 and ξ_2. Eliminating A and B from these two equations we obtain

$$N(\xi_2) - N(\xi_1) + \mu(\epsilon_2 - \epsilon_1)\pi i = m\pi, \tag{9.4}$$

where m is an integer and ϵ_1 and ϵ_2 are the ϵ values associated with ξ_1 and ξ_2, respectively. Since all terms of this equation except the last term on the left are real, ϵ_1 must be equal to ϵ_2, that is, ξ_1 and ξ_2 must be of the same sign, which requires $|q|$ to be greater than unity. Therefore, none of the neutral perturbations can propagate with the mean wind of the layer.

Now we shall show that for $q < -1$, m is positive whereas for $q > 1$, m is negative, and that m is a monotone function of q.

To prove the first statement we make use of the Wronskian $K_{n,-n}$ of $I_n(\xi)$ and $I_{-n}(\xi)$. From (9.1) and (9.2) we find

$$K_{n,-n} = -2i\xi R^2 \, dN/d\xi.$$

Since $K_{n,-n}$ is a constant, it must be equal to the value at $\xi = 0$ where $\xi \, dN/d\xi = \frac{1}{2}\mu$. Thus we have

$$\frac{dN}{d\xi} = \frac{\mu}{\xi} \frac{R^2(0)}{R^2(\xi)}. \tag{9.5a}$$

Equation (9.4) is therefore equivalent to

$$m\pi = \mu R^2(0) \int_{\xi_1}^{\xi_2} \frac{d\xi}{\xi R^2(\xi)}. \tag{9.5b}$$

This equation shows that m is positive when both ξ_1 and ξ_2 are positive ($q < -1$) and m is negative when ξ_1 and ξ_2 are negative ($q > 1$).

Since ξ_1 and ξ_2 are linear functions of q, (9.5b) determines the permissible values of q. We shall show that for each integral value of m, with the exception of zero, there is one and only one permissible q which we denote by q_m.

The fact that there is no solution for $m = 0$ can be seen directly from Eq. (2.12) because $m = 0$ corresponds to an infinite q or C. The last two terms disappear and w is then an exponential function of z and therefore cannot satisfy the two boundary conditions.

To show that m is a monotonic function of q, we differentiate (9.5b) with respect to q and obtain

$$\pi \frac{dm}{dq} = 2k^2 h\mu R^2(0) \int_{z_1}^{z_2} \left[1 + \frac{2\xi R'(\xi)}{R(\xi)}\right] \frac{dz}{\xi^2 R^2}. \tag{9.6}$$

As has been pointed out before, R has no zeros. Similarly, R' has no zeros and is therefore of one sign. In fact, both $R(\xi)$ and $R'(\xi)$ behave as $e^{\frac{1}{2}\xi}$ for positive ξ and as $e^{-\frac{1}{2}\xi}$ for negative ξ, and $\xi R'(\xi)$ is, therefore, always positive. Thus, the integral on the right side of (9.6) is positive and m is a monotonic increasing function of q.

Thus we have arrived at the following results: When n is purely imaginary, the characteristic equation (4.4) has an infinite number of simple positive real zeros and a corresponding infinite number of negative real zeros for any given wavenumber k_1, with the positive eigenvalue g_m forming a sequence such that

$$\infty > q_m > q_{m-1} > \cdots > q_1 = 1 + \epsilon > 1,$$

while the negative eigenvalues q_{-m} are given by

$$q_{-m} = -q_m, \quad \text{with} \quad m = 1, 2, 3, \cdots .$$

This result has been put forward by Eliassen, Hoiland, and Riis[9] without proof.

Further, the eigenvalue cannot be complex for this case as has already been shown by the integral relations (3.3b).

When the shearing layer extends to infinity, m can only be positive and the first set of eigenvalues is absent.

The corresponding set of eigenfunctions is a complete set and therefore any initial disturbance can be represented by a superposition of these infinite set of neutral waves, as has been shown by Eliassen, Hoiland, and Riis.

We notice the complete correspondence between this set of eigensolutions and the eigensolutions represented by (2.14) for a constant mean flow. It appears that these eigenfunctions merely represent the stable gravitational waves as modified by the shearing mean flow. The condition is completely stable as in the case of a constant mean flow.

The treatment of arbitrary initial perturbations for this case can be carried out in the same manner as in Sec. VIII. Since n is purely imaginary, the behavior of the density perturbation for large t is $t^{-\frac{1}{2}}$, and therefore will disappear gradually even without the dissipative damping factor.

X. EFFECT OF DENSITY ATTENUATION WITH HEIGHT

When the perturbation is assumed to be independent of $y(k_2 = 0)$, the term involving β in (4.1) can no longer be neglected outright because then the value of $\beta/(k^2 + \frac{1}{4}\beta^2)^{\frac{1}{2}}$ ranges from zero (for very short waves) to 2 (for very long waves). We now examine briefly the effect of this term on the stability of the perturbation.

When the term with β is included in (4.1), the solution is given by

$$W = W_{K,n}(\zeta) - \frac{W_{K,n}(\zeta_2)}{W_{-K,n}(\zeta_2)} W_{-K,n}(\zeta), \qquad (10.1)$$

where $\zeta = 2\xi$, $K = -\frac{1}{2}\beta(k^2 + \frac{1}{4}\beta^2)^{-\frac{1}{2}}$, and $W_{K,n}(\zeta)$ and $W_{-K,n}(\zeta)$ are Whittaker's functions. When $K = 0$, these functions reduce to Bessel functions multiplied by $\zeta^{\frac{1}{2}}$. Therefore, β has no effect on the very short waves.

To examine the effect of β on the long waves, we shall again treat separately the problem for an infinite layer and that of a layer bounded between two planes.

(i) Layer of infinite depth. For this case the second term in (10.1) disappears and the problem reduced to finding the zeros of $W_{K,n}(\zeta)$.

For real n, $W_{K,n}(\zeta)$ has complex zeros only when the parameter $b(=1 + 2n)$ is greater than $a + 2$, where $a = n + \frac{1}{2} - K$. Thus, the criterion is given by

$$n > 1.5 - K. \qquad (10.2)$$

Since $K = -1$ for $k_1 = 0$, we find the criterion for very long waves is $n = 2.5$. Thus in an infinite atmosphere β has a stabilizing effect on the long waves.

(ii) Layer bounded between two horizontal planes. To analyze the effect of β on the very long waves, we make use of the power series expansions of the Whittaker functions. For convenience, we use the functions $M_{K,n}(\zeta)$ and $M_{K,-n}(\zeta)$, given by

$$M_{K,\pm n}(\zeta) = \zeta^{\frac{1}{2}\pm n}\left[1 - \frac{K}{1 \pm 2n}\zeta + \cdots\right]. \qquad (10.3)$$

Substituting these functions in the boundary conditions and eliminating the arbitrary constants, we find that the criterion of instability is again given by $n = 1$. Therefore, β has no effect on the stability of the very long waves for a layer of finite depth.

XI. DISCUSSIONS OF RESULTS AND APPLICATION TO CLOUD STREETS FORMATION

The analysis above shows that the stability properties of plane Couette flow in a stratified atmosphere for three-dimensional perturbations can most conveniently be described in terms of a modified Richardson number $\bar{J} = (1 + k_1^{-2}k_2^2)gS_Z U_z^{-2}$, a vectorial dimensionless wavenumber $\alpha = h(k_1^2+k_2^2)^{\frac{1}{2}}$ and a dimensionless amplification factor $\sigma = \alpha C_i U_z^{-1}$, where $S_Z = \theta_0^{-1}\ \partial\theta_0/\partial z$, θ_0 being the undisturbed potential temperature, U_z is the vertical shear, h is the half-depth of the shear layer, and k_1 and k_2 are the wavenumbers in the windward and transversal directions. When the parameter $\bar{J}$ is above a critical value $\bar{J}_0(\alpha)$, the flow is stable, whereas when $\bar{J}$ is below $\bar{J}_0$, the perturbations grow exponentially with time. For the case of a bounded atmospheric layer, $\bar{J}_0$ has the value -0.75 for $\alpha = 0$, and it decreases gradually to -2 until $\alpha = 1.2$, after which it remains constant.

The region of stability ($\bar{J} > \bar{J}_0$) is composed of two regions, denoted as III and IV in Fig. 4, the demarcation being $\bar{J} = 0.25$. In region IV (i.e., $\bar{J} > 0.25$), there exists a complete spectrum of modified, stable internal gravitational waves, whereas in region III no eigensolutions exist and the initial perturbations must be represented by a superposition of the continuous spectrum of singular solutions. Such initial thermal and velocity perturbations behave like $t^{n-0.5} \exp[-\nu t(k^2 + \pi^2/d_1^2)]$ and $t^{n-1.5} \exp[-\nu t(k^2 + \pi^2/d_1^2)]$, respectively, for large t, and therefore will be damped out gradually by the dissipative forces. However, it must be pointed out that for $0.5 < n < 1$, the initial thermal perturbations may have a tendency to grow momentarily, even though ultimately they will be damped out by the dissipative forces if no finite amplitude instability is being created.

For values of $\bar{J}$ below the critical value $\bar{J}_0$, the perturbations grow exponentially with time if the amplification rate $k_1 C_i = k_1 \sigma U_z/k$ is larger than the dissipation rate $\nu(k^2 + \pi^2/d_1^2)$. One of the most interesting results obtained in this study is that this region of instability is also divided into two separate regions, designated as regions I and II in Fig. 4, in which the perturbations behave quite differently. The demarcation of these two regions is a sloped curve on which α increases as $-\bar{J}$ increases. In region I, which is for large-scale, convective-type perturbations, all the disturbances are anchored to the midlevel mean motion while their growth rates vary with the wavenumber and have a maximum at a certain intermediate α for any given $\bar{J}$. On the other hand, in region II, which is for small-scale, gravity-wave-type perturbations, the growth rate is almost independent of the wavenumber while they all move relative to the midlevel mean wind. The phase speed of these unstable, small-scale perturbations decreases with decreasing $\bar{J}$ but increases with increasing α, and the limiting phase speed for the shortest disturbance is $\pm U_h$. The presence of this region shows that the concept of exchange of stability is invalid for these small-scale perturbations.

The specialization of our results for three-dimensional perturbations to two-dimensional perturbations in the wind direction is simple; we achieve this by setting $k_2 = 0$. Therefore, the results then refer to $\bar{J} = gS_Z U_z^{-2} = J$, the ordinary Richardson number, $\alpha = k_1 h$, and $\sigma = k_1 C_i U_z^{-1}$. It is then clear that such two-dimensional perturbations can become unstable only when the stratification factor is below a finite negative limit such that J is below the critical value $J_0(\alpha)$ which starts with the value -0.75 at $\alpha = 0$ and decreases to -2.0 when $\alpha > 1.2$.

From this result we immediately arrive at the most important conclusion that, when the stratification is unstable, we must take the cross-wind variation into consideration by allowing a nonzero k_2. This is more so if the Richardson number is only slightly negative, e.g., $0 > J > -2$. Under such conditions, roll-type ($k_1 = 0$) convective motions with their axes parallel to the mean wind are bound to develop because they are not inhibited by the stabilizing effect of the plane Couette flow, since all the $U\,\partial/\partial x$ terms disappear in the pertinent equations. This is undoubtedly the physical reason of the predominance of longitudinal roll clouds in the earth's atmosphere and in laboratory experiments when a shearing flow is present and when the stratification is unstable.

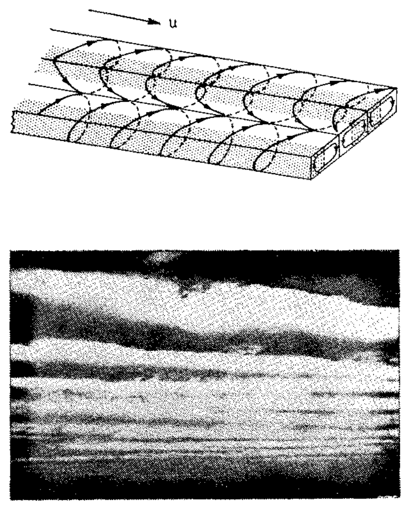

FIG. 5. (a) Schematic representation of the helicoidal particle trajectories in longitudinal convection rolls and stream line patterns in the perpendicular cross section. Clouds form in shaded ascending regions and clear sky in descending regions. (b) Observed longitudinal alto-cumulus cloud rolls, mean wind parallel to cloud from left to right.

However, the question arises as to under what conditions the motion will become truly three-dimensional, that is, neither k_2 nor k_1 is zero, and what is the preferred value of k_2/k_1 for a given Richardson number.

In order to interpret our results incorporated in Figs. 1–4 for these three-dimensional disturbances, we must obtain the actual growth rate p_r from σ and see how it depends upon S_Z, U_z, k_1, and k_2. From the definition of σ we find that the nonviscous part of the growth rate is $k_1 C_i = k_1 \sigma U_z/k$. The dissipative correction $-\nu(k^2 + \pi^2/d_1^2)$ need not be considered here. From the solutions in Sec. VI we find the maximum values of $k_1 C_i(-gS_Z)^{-\frac{1}{2}}$ for $-\bar{J} = 2$, 6 and 12 are 0.22, 0.48 and 0.60, respectively, corresponding to $\alpha = \alpha_m = 0.803$, 1.35, 1.70. These values show that both $k_1 C_i(-gS_Z)^{-\frac{1}{2}}$ and α_m are increasing with increasing $(-\bar{J})$ and that $k_1 C_i(-gS_Z)^{-\frac{1}{2}}$ approaches the maximum value of 1 asymptotically as $(-\bar{J})$ approaches infinity. Thus, the most preferred motion corresponds to the largest $(-\bar{J})$. Now $-\bar{J}$ can be large when $-J$ itself is large. Then the inhibiting effect of the Couette flow on the wave motions in the x direction is small and the perturbations will be truly three-dimensional. On the other hand, for small negative J, $(-\bar{J})$ can be large only when k_1 is much smaller than k_2. Thus we conclude that, when $(-J)$ is positive but small, longitudinal rolls will be the dominant form of convection, whereas when $(-J)$ is large, these long rolls will break into separate elongated cells by the wave motions in x direction, because then such longitudinal wave perturbations will also be excited.

Since the formations of longitudinal cloud bands in the earth's atmosphere is always associated with the presence of vertical shear of the mean wind, the present theory offers a dynamical explanation of the origin of these clouds. As a qualitative comparison between the theory and observation, we have plotted in Fig. 5(a) schematically the helicoidal particle trajectories given by the theory. The region of ascending motion has been shaded to indicate cloud formation when water vapor is present. Figure 5(b) shows a series of actual longitudinal alto-cumulus rolls which are associated with a shearing wind in the cloud direction.

Another illustration of the theoretical prediction is given in Fig. 6, which shows the schematic distribution of cumuliform clouds near the wave trough in the easterlies, constructed from time-lapse photographs by Malkus and Ronne.[4] These distributions show very clearly the breakup of the longitudinal rolls by the longitudinal wave perturbations. In particular, we notice the formation of much larger and deeper clouds along the lines of ascent of the longitudinal wave perturbations on the right of the trough line, and the dissolving of the clouds along the lines of descent.

These illustrations show that there is no doubt that the present theory offers a consistent physical explanation to such cloud formations; they are simply the products of unstable modes of perturbations initiated by the combined effects of shear flow and

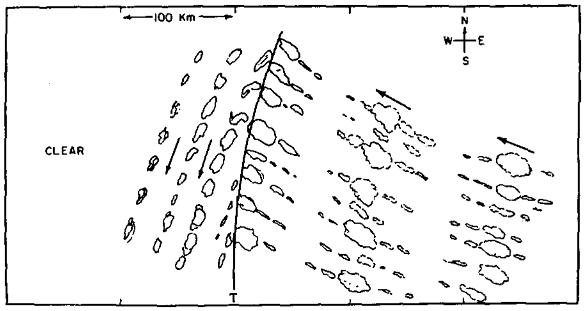

FIG. 6. Schematic distribution of clouds in relation to a trough-line in the Easterlies, constructed from time-lapse photographs (after Malkus and Ronne).

unstable stratification. However, quantitative comparison between theory and observations awaits further refinements of both types of investigation.

A word of caution seems to be appropriate at this point of our investigation since we have not as yet analyzed the possible influence of a variable U_z, whereas Kuettner's[2] observations do indicate that such variations are present. However, Avsec's[1] results definitely show that the influence of U_{zz} is small for the formation of longitudinal rolls in his experiments.

Another point worth mentioning is that for small negative J, transversal rolls may also appear during the first moment of the creation of unstable stratification in shear flow, either due to the initial growth of the thermal longitudinal wave perturbations or due to forced motion.[18] However, as has already been pointed before, such transversal rolls will not last under the assumed conditions because they are subject to the influence of the dissipative forces which introduces an exponential damping effect.

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[18] A. Graham, Phil. Trans. Roy. Soc. (London) **A232**, 285 (1933).