Comparison of quasilinear diffusion coefficients for parallel propagating whistler mode waves with test particle simulations

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[1] We present a comparison between the classical quasilinear diffusion coefficients and those calculated using a general test particle code. The trajectories of a large number of electrons are followed as they traverse a numerically-constructed, broadband, small-amplitude wave field, using a general relativistic test particle code. The change in each electron's pitch angle and energy is shown to be stochastic and the resulting diffusion of the entire population is found to be in excellent agreement with quasilinear theory. We also demonstrate that the diffusion coefficients presented by Summers, derived specifically for parallel propagating waves, are a factor of two larger than the test particle results if the power spectral density is one-sided ($\omega > 0$). Our results demonstrate the general validity of using quasilinear theory to describe the effects of broadband small amplitude waves on radiation belt electrons. **Citation:** Tao, X., J. Bortnik, J. M. Albert, K. Liu, and R. M. Thorne (2011), Comparison of quasilinear diffusion coefficients for parallel propagating whistler mode waves with test particle simulations, *Geophys. Res. Lett.*, *38*, L06105, doi:10.1029/2011GL046787.

1. Introduction

[2] Quasilinear diffusion theory has been widely used in modeling radiation belt dynamics [e.g., *Lyons and Thorne*, 1973; *Kennel and Petschek*, 1966; *Horne et al.*, 2005], which is a crucial part of space weather specification and prediction. The quasilinear theory describes interactions between charged particles and small amplitude broadband waves in terms of a diffusion equation and diffusion coefficients. The general form of the quasilinear diffusion coefficients, written in velocity space ($v_\perp$, $v_\parallel$), with a uniform background magnetic field and a distribution of plasma waves with arbitrary wave normal angles was described by *Kennel and Engelmann* [1966] for nonrelativistic particles and by *Lerche* [1968] for relativistic particles. Here $v_\perp(v_\parallel)$ is the velocity component perpendicular (parallel) to the background magnetic field. *Lyons* [1974] expressed the diffusion coefficients of *Kennel and Engelmann* [1966] in terms of pitch angle (α) and energy (E), which was widely adopted subsequently by the radiation belt community. Several numerical codes [*Glauert and Horne*, 2005; *Albert*, 2005; *Shprits and Ni*, 2009; *Xiao et al.*, 2009, 2010] have recently been developed, following *Lyons* [1974], to calculate the quasilinear diffusion coefficients and model radiation belt dynamics. We will use *Kennel and Engelmann* [1966] to represent quasilinear diffusion theory discussed above, even though it is only strictly applicable for non-relativistic electrons.

[3] The quasilinear diffusion coefficients for parallel propagating waves have been reduced to a set of closed analytical forms for computational convenience [*Summers*, 2005]. However, as later noted by *Albert* [2007], the starting point of this analysis, namely equation (1) of *Summers* [2005] has an extra factor of 2 compared with that of *Kennel and Engelmann* [1966], and correspondingly the diffusion coefficients of *Summers* [2005] are two times larger than those of *Kennel and Engelmann* [1966], *Lyons* [1974], *Glauert and Horne* [2005] and *Albert* [2005]. This difference in the value of the diffusion coefficients has serious implications because the expressions from both *Summers* [2005] and *Kennel and Engelmann* [1966] are widely used, even though the "factor-of-two" discrepancy has not been resolved.

[4] *Liu et al.* [2010a, 2010b] used a test particle simulation to calculate diffusion coefficients of interactions between relativistic electrons and electromagnetic ion cyclotron (EMIC) waves generated from a hybrid simulation. Good agreement has been found between the test particle results and the quasilinear theory of *Summers* [2005] (but see a discussion in Section 3). As an independent and complementary approach, we use a whistler wave field created from a summation of plane waves in a test particle simulation both to verify the validity of the quasilinear theory and to investigate the factor of 2 difference between *Summers* [2005] and *Kennel and Engelmann* [1966]. In the course of the analysis, we also attempt to clarify several basic concepts that can potentially cause confusion in the calculation of the diffusion coefficients.

2. Test Particle Simulation

[5] We use a homogeneous background magnetic field $\mathbf{B}_0 = B_0\mathbf{e}_z$ in our simulation, to be consistent with basic formulation of the quasilinear theory by both *Kennel and Engelmann* [1966] and *Summers* [2005]. Test particle trajectories are calculated by solving the full relativistic Lorentz equation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\boldsymbol{p}}{\gamma m}, \quad (1)$$

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\left[\boldsymbol{E}_\mathrm{w} + \frac{\boldsymbol{p}}{\gamma m} \times (\boldsymbol{B}_\mathrm{w} + \boldsymbol{B}_0)\right], \quad (2)$$

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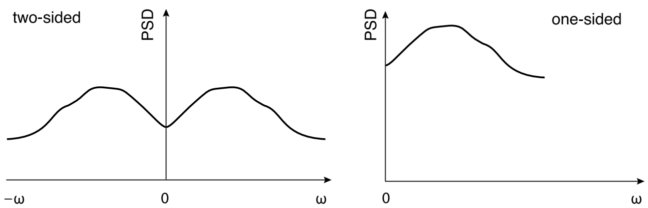

Figure 1. Illustration of the concepts of the (left) two-sided wave power spectral density and (right) one-sided power spectral density.

using a fifth order Cash-Karp Runga-Kutta method [*Press et al.*, 2002, pp. 721–722]. The wave field is composed of a summation of parallel-propagating whistler waves, represented by

$$\boldsymbol{B}_{\mathrm{w}} = \sum_{j=1}^{N_{\mathrm{w}}} B_{xj}^{\mathrm{w}} \cos\Phi_j \mathbf{e}_x - B_{yj}^{\mathrm{w}} \sin\Phi_j \mathbf{e}_y, \quad (3)$$

$$\boldsymbol{E}_{\mathrm{w}} = \sum_{j=1}^{N_{\mathrm{w}}} -E_{xj}^{\mathrm{w}} \sin\Phi_j \mathbf{e}_x - E_{yj}^{\mathrm{w}} \cos\Phi_j \mathbf{e}_y, \quad (4)$$

with the instantaneous wave amplitudes of the jth component calculated using cold plasma wave theory and a chosen B_{yj}^{w} [*Tao and Bortnik*, 2010; *Stix*, 1992]. Here the wave phase angle of each component $\Phi_j \equiv k_j z - \omega_j t + \phi_{0j}$ and the wave frequency $\omega_j = \omega_{\min} + j\Delta\omega, j = 1, 2, \cdots, N_{\mathrm{w}}$. The initial wave phase angle ϕ_{0j} is randomly chosen between 0 and 2π, and the wave number k_j is calculated from ω_j using cold plasma theory. To simulate a smooth wave power spectral density, we choose $\omega_{\min}$ to be a multiple of $\Delta\omega$. We use an ensemble of electrons with identical initial pitch angle and energy values, but having random initial gyrophases (the angle between $\boldsymbol{v}_\perp$ and $\mathbf{e}_x$) and z coordinates, and calculate the wave-induced changes of their pitch angle and energy. The test particle diffusion coefficients are calculated from $D_{\alpha\alpha}^{\mathrm{TP}} \equiv \langle\Delta\alpha^2\rangle/2\Delta t$ and $D_{EE}^{\mathrm{TP}} \equiv \langle\Delta E^2\rangle/2\Delta t$, where $\langle\cdots\rangle$ denotes averaging over all electrons, $\Delta\alpha \equiv \alpha - \langle\alpha\rangle$ and $\Delta E \equiv E - \langle E\rangle$.

[6] To calculate the theoretical quasilinear diffusion coefficients (D^{QL}), it is extremely important to specify the wave power spectral density correctly. A potential source of error involves the two commonly used ways of specifying the wave power spectral density, as illustrated in Figure 1. In the quasilinear theory of, e.g., *Kennel and Engelmann* [1966], the wave power spectral density is assumed to be two-sided ($P_{\mathrm{T}}(\omega)$ with $-\infty < \omega < \infty$) for mathematical convenience. This is reflected in the condition $\omega(-k) = -\omega(k)$, which is explicitly used in the derivation of the theory. It is important to note that if a two-sided power spectral density is used, the R-mode wave is represented by $B_x - \mathrm{i}B_y$ if $\omega > 0$ and $B_x + \mathrm{i}B_y$ if $\omega < 0$ [*Kennel and Engelmann*, 1966; *Stix*, 1992, p. 499]. In addition, the resonance condition is

$$\omega - k_\parallel v_\parallel = n\Omega/\gamma, \; n = 0, \pm1, \pm2, \cdots, \quad (5)$$

with $k_\parallel$ being the component of k parallel to ambient magnetic field, Ω the charged particle's nonrelativistic gyrofrequency, and γ the relativistic factor. If equation (5) is satisfied by ($\omega^*(k)$, $k_\parallel^*$, n^*), then ($\omega^*(-k)$, $-k_\parallel^*$, $-n^*$) is also a solution because $\omega^*(-k) = -\omega^*(k)$. For example, if a two-sided wave power spectral density is used in considering electrons interacting with unidirectional parallel propagating whistler waves, both the $n = -1$ and $n = 1$ resonances are needed in the calculation of diffusion coefficients, and their contributions to the final diffusion coefficients are equal to each other.

[7] On the other hand, a one-sided power spectral density ($P_{\mathrm{O}}(\omega)$ with $0 \le \omega < +\infty$) is widely used in the calculation of diffusion coefficients, especially since it can be readily related to observed spectra [e.g., *Lyons*, 1974]. However, for real input signals, the one-sided wave power spectral density is larger than the two-sided power spectral density by a factor of 2; i.e., $P_{\mathrm{O}}(\omega) = P_{\mathrm{T}}(-\omega) + P_{\mathrm{T}}(\omega) = 2P_{\mathrm{T}}(\omega)$ with $\omega > 0$ [*Press et al.*, 2002, pp. 503–504]. The equivalence of the two choices of wave power spectral density in calculating D^{QL} can be inferred from the fact that $\mathsf{D}^{\mathrm{QL}} \propto P_{\mathrm{T}}(-\omega) + P_{\mathrm{T}}(\omega)$, $\omega > 0$, because resonances from both ω and $-\omega$ contribute to D^{QL}, and this is equivalent to $\mathsf{D}^{\mathrm{QL}} \propto P_{\mathrm{O}}(\omega)$, where only positive ω is considered. While the choice of wave power spectral density is not explicitly stated by *Summers* [2005], it is interpreted here as being one-sided (but see discussions below). The theoretical quasilinear diffusion coefficients are calculated using equations (17), (18), and (27) of *Summers* [2005], and are further divided by 2 to give diffusion coefficients equivalent to *Kennel and Engelmann* [1966], as discussed above. We will denote diffusion coefficients from *Kennel and Engelmann* [1966] by D^{QL} and from *Summers* [2005] by $\mathsf{D}^{\mathrm{QL,S}}$.

3. Results and Discussion

[8] In the following test particle calculations, we choose $B_0 = 1.4 \times 10^{-7}$ T to represent the geomagnetic field at geocentric distance $r \sim 6R_{\mathrm{E}}$, R_{E} being the Earth radius, and a cold electron density of $n_e = 10\ \mathrm{cm}^{-3}$, representative of the plasmatrough. In equation (3), we use $N_{\mathrm{w}} = 100$, $\omega_{\min} = 0.2\Omega_e$, $\omega_{\max} = 0.4\Omega_e$, and $B_{yj}^{\mathrm{w}} = 1$ pT for all j to create a flat spectrum between $\omega_{\min}$ and $\omega_{\max}$. Here Ω_e is the nonrelativistic electron cyclotron frequency. The mean squared amplitude of the broadband wave is $\sqrt{\overline{B_{\mathrm{w}}^2}} = \sqrt{N_{\mathrm{w}}} B_{yj}^{\mathrm{w}} = 10$ pT. The bandwidth and amplitudes are chosen so that they are consistent with the assumptions of quasilinear theory. We have confirmed numerically (not shown here) that the distance between two adjacent resonances is smaller than the sum of the half-widths of the two resonance islands, resulting in stochastic behavior of electrons [*Lichtenberg and Lieberman*, 1983, pp. 245–265].

[9] We calculate trajectories of 400 electrons with the same initial pitch angle and energy in each run, with the initial energy chosen to satisfy the resonance condition (equation (5)) with waves at about $\omega = 0.3\Omega_e$. A sample calculation is shown in Figure 2, for electrons with initial pitch angle $\alpha = 40°$ and $E = 9.313$ keV. The changes of α and E of five randomly selected electrons are shown in the first row to demonstrate that their behavior is indeed stochastic. In Figure 2 (bottom), we compare the changes of $\langle\Delta\alpha^2\rangle$ and $\langle\Delta E^2\rangle$ obtained from test particle calculations with quasilinear theory results $\langle\Delta\alpha^2\rangle = 2D_{\alpha\alpha}^{\mathrm{QL}}\, t$ (and similarly for $\langle\Delta E^2\rangle$), obtained from *Kennel and Engelmann* [1966] (blue lines) and *Summers* [2005] (red lines). Notably the quasilinear results of *Kennel and Engelmann* [1966] are in good agreement with the test particle calculations, while the diffusion coefficients of *Summers* [2005] are a factor of 2 higher.

[10] It should be noted that *Liu et al.* [2010a, 2010b] used parallel EMIC wave fields generated by a hybrid code and a test particle simulation to calculate pitch angle diffusion coefficients. They compared their results with that of *Summers* [2005] and found good agreement. However, the quantity $W(k)$ in equation (5) of *Liu et al.* [2010a] is the wave power spectral density of $B_L \equiv (B_x + iB_y)/\sqrt{2}$ for $\omega > 0$, which is half of the corresponding $W(k)$ given by equations (10) and (11) of *Summers* [2005] (not explicitly stated by *Liu et al.* [2010a]). Thus, the results of *Liu et al.* [2010a, 2010b] also agree with *Kennel and Engelmann* [1966] and our present work.

[11] The above conclusion is obtained if the wave power spectral density of *Summers* [2005] is interpreted as one-sided. However, if a two-sided power spectral density was actually employed in equation (1) of *Summers* [2005] and since only positive frequencies are actually considered, the results of *Summers* [2005] would be identical to those of *Kennel and Engelmann* [1966]. This is because $\mathrm{D}^{\mathrm{QL,S}} \propto 2P_T(\omega)$, $\omega > 0$, is essentially the same as $\mathrm{D}^{\mathrm{QL}} \propto P_O(\omega)$. Thus the correctness of diffusion coefficients of *Summers* [2005] depends on the appropriate interpretation of the wave power spectral density, which is not explicitly stated by *Summers* [2005].

[12] In order to further compare the test particle scattering against quasilinear theory, we repeat the runs above for a set of different initial pitch angles. The initial energy of electrons is correspondingly changed so that they initially resonate with waves at $\omega = 0.3\Omega_e$. The test particle diffusion coefficient $D_{\alpha\alpha}^{\mathrm{TP}}$ (or D_{EE}^{TP}) is obtained by fitting $y = A + Bt$ to the line of $\langle\Delta\alpha^2\rangle$ (or $\langle\Delta E^2\rangle$) as a function of t, and $D_{\alpha\alpha}^{\mathrm{TP}}$ (or D_{EE}^{TP}) $= B/2.0$. We have checked that $|A|$ is close to 0 and less than B by at least three orders of magnitude for all runs. The comparison between D^{QL} and D^{TP} is shown in Figure 3, where the initial pitch angles range from 10° to 80°, in a step of 10°. It can be seen that D^{QL} is in excellent agreement with D^{TP}, which demonstrates that quasilinear theory provides an excellent theoretical framework for describing the effects of resonant interactions between electrons and broadband small amplitude whistler waves.

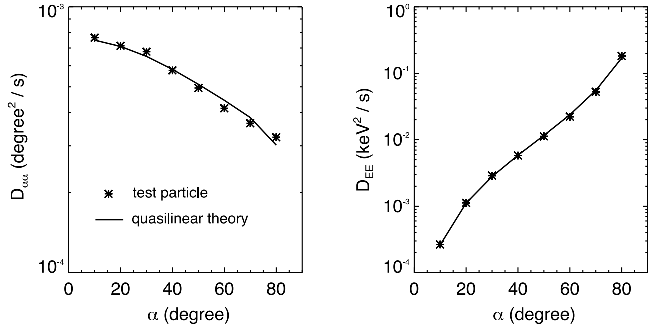

Figure 3. The comparison between test particle diffusion coefficients (stars) and theoretical diffusion coefficients (solid lines) of *Kennel and Engelmann* [1966]. (left) $D_{\alpha\alpha}$ and (right) D_{EE}.

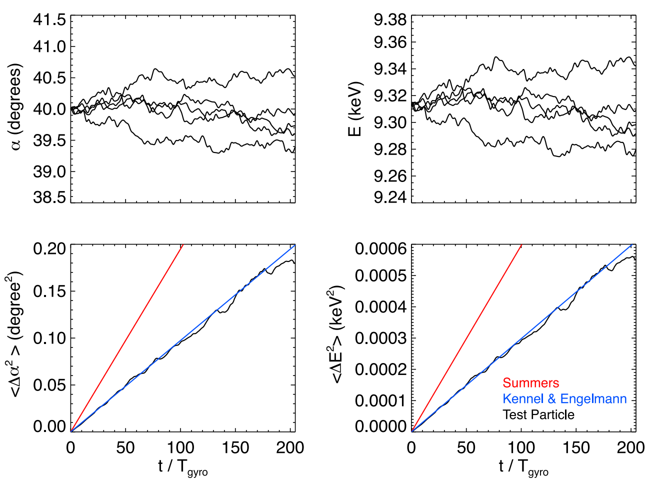

Figure 2. Results of a run of resonant interactions between an ensemble of electrons with initial pitch angle $\alpha = 40°$ and whistler waves specified in the text. (top) Changes of (left) pitch angle and (right) energy of five randomly selected electrons. (bottom) Changes of (left) $\Delta\alpha^2$ and (right) ΔE^2 averaged over all 400 electrons. Also plotted are changes of $\langle\Delta\alpha^2\rangle$ and $\langle\Delta E^2\rangle$ calculated using quasilinear diffusion coefficients of *Kennel and Engelmann* [1966] (blue) and *Summers* [2005] (red).

4. Summary

[13] Using a general, fully relativistic Lorentz test particle code with minimal assumptions, we showed that the scattering induced by a broadband, incoherent, small-amplitude wave field is stochastic and is in excellent agreement with the diffusion coefficients derived from the quasilinear diffusion theory of *Kennel and Engelmann* [1966]. We show that the factor of 2 difference in the formula of *Summers* [2005] is not needed when calculating quasilinear diffusion coefficients, if the wave power spectral density of *Summers* [2005] is interpreted as one-sided.

[14] Having developed the technique to make a direct comparison between test particle scattering calculations and the predictions of quasilinear theory has important implications. First, as demonstrated in this paper, it is possible to independently verify the absolute value of the diffusion coefficient for a set of identical initial conditions. Second, it is possible to investigate the qualitative nature of the scattering, be it diffusive, advective, or a combination of the above. Third, it is possible to investigate the effects of processes that fall outside the descriptive domain of quasilinear theory, such as nonlinear and non-resonant scattering (both of which have been shown to play a role in radiation belt dynamics), and to cast these processes in the framework of Fokker-Planck theory for use in transport codes. These investigations will be the subject of future work.

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