Observability of Vortex Flows

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Observability of Nonlinear Dynamics

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \\ x(0) &= x^0 \\ x \in \mathbb{R}^n, & \quad y \in \mathbb{R}^p, \quad p \leq n \end{aligned}$$

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$y(0:T)$ **is the curve**

$$t \mapsto y(t)$$

for $0 \leq t < T$

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is locally one to one for any $T > 0$.

Differentiation

The exterior derivative of the function h^i is the one form

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We can iterate this operation

$$\begin{aligned} L^0_f(h^i)(x) &= h^i(x) \\ L^r_f(h)(x) &= \frac{\partial L^{r-1}h^i}{\partial x_j}(x)f_j(x) \end{aligned}$$

for $r = 1, 2, \ldots$

Observability Rank Condition

The observed system satisfies the observability rank condition (ORC) at x if

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The observed system satisfies the observability rank condition if it satisfies the ORC at every $x \in \mathbb{R}^n$.

Interpretation of Observability Rank Condition

$$\begin{aligned}\dot{x} &= f(x)\\ y &= h(x)\\ \dot{y} &= L_f(h)(x)\\ \ddot{y} &= L_f^2(h)(x)\\ &\vdots\end{aligned}$$

If the observability rank condition holds then the functions $h(x),\ L_f(h)(x),\ L_f^2(h)(x), \ldots$ distinguish neighboring points

Strong Observability Rank Condition

The observed system satisfies the strong observability rank condition (SORC) at x if the covectors

$$\left\{ dL_f^r(h)(x) : r = 0, 1, 2, \ldots, \lceil n/p \rceil - 1 \right\}$$

are linearly independent and

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The observed system satisfies the strong observability rank condition if it satisfies the SORC at every $x \in \mathbb{R}^n$.

If this is the case then we use the lowest possible derivatives of y to determine x.

Theorem

Suppose the observed system satisfies the observability rank condition then it is short time, locally observable.

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If the observed system fails to satisfy the observability rank condition on an open subset of $\mathbb{R}^n$ then it is not short time, locally observable.

If the observed system satisfies the strong observability rank condition then an extended Kalman filter is locally convergent to the true state in the absence of driving and observation noises.

Finite Dimesional Fluid

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on $I\!R^n$ and an x dependent vector field

$$u(x, \xi)$$

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The fluid satisfies the differential equation

$$\dot{\xi} \;=\; u(x, \xi)$$

Example, Vortex Flows

A single point vortex at x_1, x_2 with strength x_3 induces the flow on $\Omega = \mathbb{R}^2$

$$u(x, \xi) \;=\; \frac{x_3}{r^2} \begin{bmatrix} x_2 - \xi_2 \\ \xi_1 - x_1 \\ 0 \end{bmatrix}$$

where $r^2 = (\xi_1 - x_1)^2 + (\xi_2 - x_2)^2$

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This is an incompressible and irrotational flow with a singularity at $\xi = (x_1, x_2)$

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The flow is stationary

$$\dot{x} \;=\; 0$$

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Suppose there are $m = n/3$ vortices. The center of the i^{th} vortex is at x_{i1}, x_{i2} and it has strength x_{i3}.

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The flow is

$$u(x, \xi) \ = \ \sum_{i=1}^{m} \frac{x_{i3}}{r_i^2} \left[\begin{array}{c} x_{2i} - \xi_2 \\ \xi_1 - x_{1i} \end{array} \right]$$

where $r_i^2 = (\xi_1 - x_{i1})^2 + (\xi_2 - x_{i2})^2$

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where $r_i^2 = (\xi_1 - x_{i1})^2 + (\xi_2 - x_{i2})^2$

This is also an incompressible and irrotational flow with m singularities at $\xi = (x_{i1}, x_{i2})$ for $i = 1, \ldots, m$.

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The i^{th} vortex moves under the influence of the other $m-1$ vortices.

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The motion is

$$\begin{bmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{bmatrix} = \sum_{j \neq i} \frac{x_{j3}}{r_{ij}^2} \begin{bmatrix} x_{j2} - x_{i2} \\ x_{i1} - x_{j1} \\ 0 \end{bmatrix}$$

where $r_{ij}^2 = (x_{i1} - x_{j1})^2 + (x_{i1} - x_{j2})^2$

Another Example

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The state x are the parameters of discretization.

Eulerian Observations

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If it is the velocity, then the observed system is

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where $x \in I\!\!R^n,\ y \in I\!\!R^d$.

We could have l Eulerian observations at the points $\xi^1, \ldots, \xi^l$

$$\begin{aligned} \dot{x} &= f(x) \\ y_1 &= h_1(x) = u(x, \xi^1) \\ &\vdots \\ y_l &= h_l(x) = u(x, \xi^l) \end{aligned}$$

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Let $\boldsymbol{\xi} = \boldsymbol{\xi}(t) \in \boldsymbol{\Omega}$ be the location of a sensor moving with the fluid. We define a new state vector

$$\boldsymbol{z} = \left[\begin{array}{c} \boldsymbol{x} \\ \boldsymbol{\xi} \end{array} \right]$$

Lagrangian Observations

The extended dynamics is

$$\dot{\boldsymbol{z}} \quad = \boldsymbol{g}(\boldsymbol{z}) = \left[\begin{array}{c} \boldsymbol{f}(\boldsymbol{x}) \\ \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{\xi}) \end{array} \right]$$

Lagrangian Observations

The extended dynamics is

$$\dot{z} \;\; = g(z) = \left[\begin{array}{c} f(x) \\ u(x,\xi) \end{array}\right]$$

and the Lagrangian observation is

$$y \;\; = \;\; k(z) = \xi$$

Lagrangian Observations

If there are $\boldsymbol{l}$ Lagrangian sensors at $\boldsymbol{\xi^i} \in \boldsymbol{\Omega}$ then

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$$y_i = k_i(z) = \xi^i$$

Are Eulerian and Lagrangian Observations Equivalent?

If the flow is observable under any l Eulerian observations, is it observable under any l Lagrangian observations?

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All we know is that the center x_1, x_2 of the vortex lies on the line perpendicular to the observed velocity, $(x, 0)$, but we don't know where because we don't know the strength x_3 .

Now assume that there is one Lagrangian observation $\xi(t)$. If we take perpendiculars to $\dot{\xi}(t)$ at two different times they will intersect at the center of the vortex. Once we know the center it is easy to determine the strength.

Observability Rank Condition Revisited

Finite dimensional fluid with l Eulerian observations

$$\begin{aligned} \dot{x} &= f(x) \\ y_i &= h_i(x) = u(x, \xi^i) \end{aligned}$$

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$$\begin{aligned} \dot{x} &= f(x) \\ y_i &= h_i(x) = u(x, \xi^i) \end{aligned}$$

The first two terms of the observability rank condition are

$$\begin{aligned} dh_i(x) &= du(x, \xi^i) = \frac{\partial u}{\partial x_j}(x, \xi^i) dx_j \\ dL_f(h_i)(x) &= dL_f(u)(x, \xi^i) \\ &= \left(\frac{\partial^2 u}{\partial x_j \partial x_s}(x) f_s(x) + \frac{\partial u}{\partial x_s}(x) \frac{\partial f_s}{\partial x_j}(x) \right) dx_j \end{aligned}$$

Observability Rank Condition Revisited

Extended finite dimensional fluid with l Lagrangian observations

$$\begin{aligned} \dot{z} &= g(z) = \begin{bmatrix} f(x) \\ u(x,\xi^1) \\ \vdots \\ u(x,\xi^l) \end{bmatrix} \\ y_i &= k_i(z) = \xi^i \end{aligned}$$

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Let d_z be the exterior differentiation operator in the z variables, i.e.,

$$d_z k(z) = \frac{\partial k}{\partial x_j}(x,\xi^1,\ldots,\xi^k)dx_j + \frac{\partial k}{\partial \xi^i_j}(x,\xi^1,\ldots,\xi^k)d\xi^i_j$$

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$$d_z k_i(z) = \begin{bmatrix} d\xi_1^i \\ d\xi_2^i \end{bmatrix}$$

These one forms span the extra dimensions of the extended system.

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The first three terms of the ORC

$$d_z k_i(z) = \begin{bmatrix} d\xi^i_1 \\ d\xi^i_2 \end{bmatrix}$$

These one forms span the extra dimensions of the extended system.

$$\begin{aligned} d_z L_g(k_i)(z) &= d_z u(x,\xi^i) = \frac{\partial u}{\partial x_j}(x,\xi^i) dx_j \\ &\quad \text{mod}\left\{d\xi^1,\dots,d\xi^m\right\} \end{aligned}$$

Modulo $d_z k_i$, these one forms span the same dimensions as dh_i.

Observability Rank Condition Revisited

$$\begin{aligned} d_z L_g^2(k_i)(z) \;&=\; \left(\frac{\partial^2 u}{\partial x_l \partial x_j}(x) f_l(x) + \frac{\partial u}{\partial x_l}(x)\frac{\partial f_l}{\partial x_j}(x)\right) dx_j \\ &\quad + \frac{\partial^2 u}{\partial x_j \partial \xi}(x,\xi^i)(x,\xi^i) dx_j, \\ &\quad \textbf{mod}\left\{d\xi^1,\ldots,d\xi^m, dL_g(\xi^1),\ldots,dL_g(\xi^m)\right\} \end{aligned}$$

But these do not span the span the same dimensions as $dL_f(h_i)$ modulo the above because of the extra term

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$$\frac{\partial^2 u}{\partial x_j \partial \xi}(x,\xi^i)(x,\xi^i) dx_j$$

Notice the extra term depends on ξ^i while the rest of $d_z L_g^2(k_i)(z)$ does not.

ORC for One Vortex Flow

One Eulerian observation at the origin.

$$\begin{aligned} dh(x) &= \frac{1}{r^4}\left[\begin{array}{l} 2x_1x_2x_3dx_1 + (x_2^2 - x_1^2)x_3dx_2 - x_2r^2dx_3 \\ (x_2^2 - x_1^2)x_3dx_1 - 2x_1x_2x_3dx_2 + x_1r^2dx_3 \end{array}\right] \\ dL_f(h)(x) &= \left[\begin{array}{c} 0 \\ 0 \end{array}\right] \end{aligned}$$

The Eulerian observed system does not satisfy the observability rank condition.

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The Eulerian observed system does not satisfy the observability rank condition.

WLOG $x_1 \neq 0,\ x_2 = 0,\ x_3 \neq 0$

$$\begin{aligned} dh(x) &= \left[\begin{array}{c} -\frac{x_3}{x_1^2}dx_2 \\ \\ -\frac{x_3}{x_1^2}dx_1+\frac{1}{x_1}dx_3 \end{array}\right] \\ dL_f^k(h)(x) &= 0, \quad k \geq 1 \end{aligned}$$

so the rank is 2. **The state dimension is** 3.

ORC for One Vortex Flow

$$dh(x) \quad = \quad \begin{bmatrix} -\frac{x_3}{x_1^2} dx_2 \\ \\ -\frac{x_3}{x_1^2} dx_1 + \frac{1}{x_1} dx_3 \end{bmatrix}$$

We can not observe changes in the initial condition that lie in the null space of dh

$$\begin{bmatrix} 1 \\ 0 \\ \frac{x_3}{x_1} \end{bmatrix}$$

The change that cannot be detected is moving the vortex away from the observer while increasing its strength.

ORC for One Vortex Flow

One Lagrangian observation momentarily at the origin. WLOG $\boldsymbol{\xi^1(t) = 0,\ x_1 \neq 0,\ x_2 = 0,\ x_3 \neq 0}$

$$
\begin{aligned}
d_z k_1(z) &= \begin{bmatrix} d\xi^1_1 \\ d\xi^1_2 \end{bmatrix} \\
d_z L_g(k_1)(z) &= \begin{bmatrix} -\dfrac{x_3}{x_1^2} dx_2 \\ \\ -\dfrac{x_3}{x_1^2} dx_1 + \dfrac{1}{x_1} dx_3 \end{bmatrix} \\
& \quad \textsf{mod} \left\{ d\xi^1, \ldots, d\xi^m \right\}
\end{aligned}
$$

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So far we have 4 linearly independent one forms.

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So far we have 4 linearly independent one forms.

The dimension of the extended state space is 5.

ORC for One Vortex Flow

The extra term in $d_z L_g^2(k)$ is

$$\left[\begin{array}{c} -\frac{2x_3^2}{x_1^4}dx_1 + \frac{x_3}{x_1^3}dx_3 \\ \\ -\frac{2x_3^2}{x_1^4}dx_2 \end{array}\right]$$

We compute the determinant

$$\left[\begin{array}{cc} -\frac{x_3}{x_1^2} & \frac{1}{x_1} \\ \\ -\frac{2x_3^2}{x_1^4} & \frac{x_3}{x_1^3} \end{array}\right] = \frac{x_3^2}{x_1^5}$$

and see that the observability rank condition is satisfied.

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One vortex flow is observable under two Eulerian observations.

If the two observations are not collinear with the center of the vortex then the center is at the intersection of the perpendiculars to the observations.

If they are collinear then a simple argument is needed to show observability.

Two Vortex Flow

Two vortex flow can be quite complicated but the motion of the centers of the vortices is relativity simple.

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Vortex one is at x_{11}, x_{12} and its strength is x_{13} .
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where $r^2 = (x_{11} - x_{21})^2 + (x_{12} - x_{22})^2$.
The distance r between the centers remains constant because each center moves perpendicular to the line between them.

Two Vortex Flow

If the magnitudes are different, $|x_{13}| \neq |x_{23}|$, the two vortices move on two concentric circles in the plane.
If the vortices are of same orientation, $x_{13}x_{23} > 0$, they stay as far away as possible on the concentric circles.

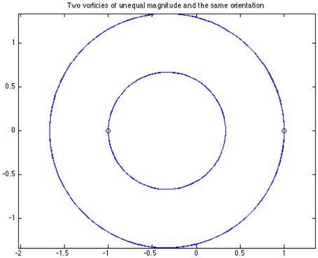
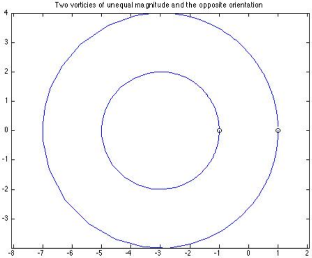


Figure: The motion of the centers of two vortices of unequal magnitudes and the same orientation. The centers are at the circles.

Two Vortex Flow

When the they are of opposite orientation, $x_{13}x_{23} < 0$, they will stay as close as possible.



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Two Vortex Flow

If the strengths are equal $x_{13} = x_{23}$, then the center will rotate around a single circle staying as far away as possible.

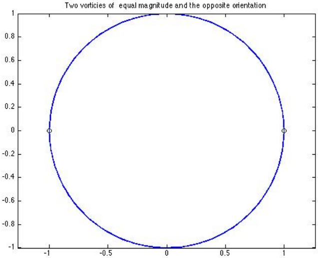

Figure: The motion of the centers of two vortices of equal magnitudes and the same orientation. The centers are at the circles.

Two Vortex Flow

If the strengths are opposite, $x_{13} = -x_{23}$, then the two centers will fly off to infinity along two parallel lines.

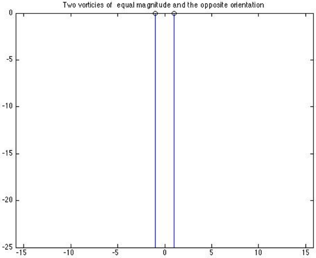

Figure: The motion of the centers of two vortices of equal magnitudes and the opposite orientation. The centers are at the circles.

Two Vortex Flow

Suppose that the strengths are not opposite $x_{13} \neq -x_{23}$ and without loss of generality the vortices start at $(x_{11}(0), x_{12}(0)) = (1, 0)$ and $(x_{21}(0), x_{22}(0)) = (-1, 0)$ then the two vortices will rotate around the point

$$\xi^c = (\xi_1^c, \xi_2^c) = (\frac{x_{13} - x_{23}}{x_{13} + x_{23}}, 0)$$

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but generally this stagnation point will rotate with the vortices remaining on the line between their centers.
The one exception is when the strengths are equal $x_{13} = x_{23}$ for then the stagnation point is the center of rotation at $(0, 0)$ and remains there.

Two Vortex Flow

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With the above assumptions, the collinear, co-rotating points are at $(\xi_1, 0)$ where ξ_1 is a root of the cubic

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When the orientations of the vortices are opposite, there is only one co-rotating point that is collinear with the vortex centers.

Eulerian Observability of Two Vortex Flow

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When the two vortices and the Eulerian observation are collinear, the rank is $\mathbf{5}$ except for a symmetric configuration where the rank is $\mathbf{3}$.

Eulerian Observability of Two Vortex Flow

A symmetric configuration is one satisfying

$$\begin{aligned} x_{21} &= -x_{11} \\ x_{22} &= -x_{12} \\ x_{23} &= x_{13} \end{aligned}$$

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Numerical calculations confirm that it is exactly 3 .

Eulerian Observability of Two Vortex Flow

Except for the symmetric case, a collinear configuration is not invariant under the dynamics if the observation is not at the center of rotation so the rank of

$$\begin{array}{r} dh(x) \\ dL_f(h)(x) \\ dL_f^2(h)(x) \end{array}$$

immediately become 6 where the SORC holds.

Eulerian Observability of Two Vortex Flow

If the observation is at the center of rotation the rank of

$$\begin{matrix} dh(x) \\ dL_f(h)(x) \\ dL_f^2(h)(x) \end{matrix}$$

remains 5 so SORC continues to not hold.

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When the two vortices and the Lagrangian observation are collinear, the rank is 7 except for the symmetric case discussed above where the rank is $5 = 8 - 3$.

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Extended Kalman Filtering

Observed dynamics

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Continuous Time Extended Kalman Filter (EKF)

$$\begin{aligned} \dot{\hat{x}}(t) &= f(\hat{x}(t)) + P(t)H'(t)\left(y(t) - h(\hat{x}(t))\right) \\ \dot{P}(t) &= F(t)P(t) + P(t)F'(t) + Q(t) \\ &\quad -P(t)H'(t)R^{-1}(t)H(t)P(t) \end{aligned}$$

where

$$\begin{aligned} F(t) &= \frac{\partial f}{\partial x}(\hat{x}(t)) \\ H(t) &= \frac{\partial h}{\partial x}(\hat{x}(t)) \end{aligned}$$

Extended Kalman Filtering

Four design parameters of the EKF,

$\hat{x}(0)$	**Initial estimate**
$P(0) \geq 0$	**Initial error covariance**
$Q(t) \geq 0$	**Driving noise covariance**
$R(t) > 0$	**Observation noise covariance**

Reduced Order Extended Kalman Filtering

Observed dynamics

$$\begin{aligned} \dot{z}_1 &= g_1(z_1, z_2) \\ \dot{z}_2 &= g_2(z_1, z_2) \\ y &= k(z_1, z_2) = z_2 \end{aligned}$$

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Reduced Order EKF (REKF)

$$\begin{aligned}\dot{\hat{z}}_1(t) &= g_1(\hat{z}_1(t), y(t)) + P_1(t)G_2'(t)\left(\dot{y}(t) - g_2(\hat{z}_1(t), y(t)\right)\\ \dot{P}(t) &= G_1(t)P(t) + P(t)G_1'(t) + Q_1(t)\\ &\quad -P(t)G_2'(t)Q_2^{-1}(t)G_2(t)P(t)\end{aligned}$$

$$\begin{aligned}G_1(t) &= \frac{\partial g_1}{\partial z_1}(\hat{z}_1(t), y(t))\\ G_2(t) &= \frac{\partial g_2}{\partial z_1}(\hat{z}_1(t), y(t))\end{aligned}$$

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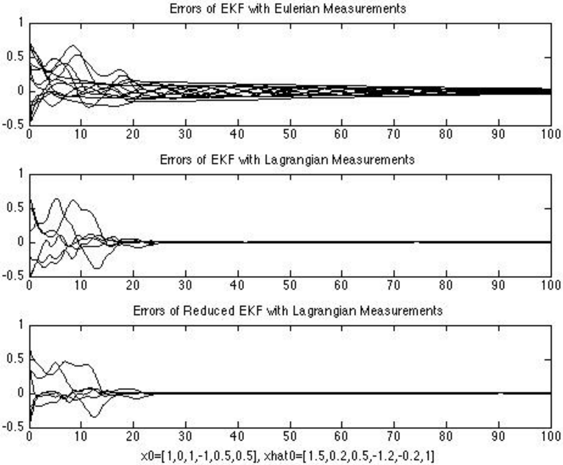

Errors of EKF with Eulerian Measurements

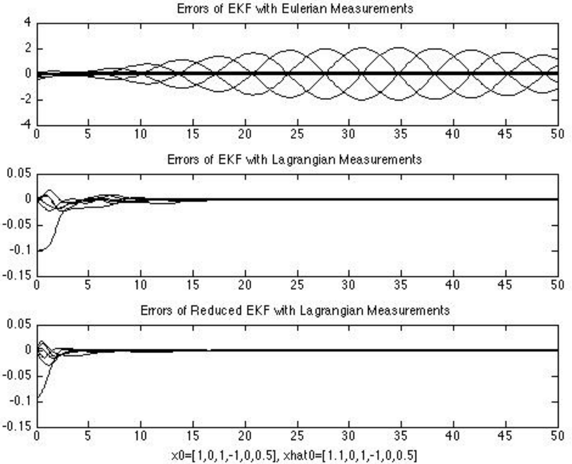
Errors of EKF with Lagrangian Measurements

Errors of Reduced EKF with Lagrangian Measurements

x0=[1,0,1,-1,0.5,0.5], xhat0=[1.5,0.2,0.5,-1.2,-0.2,1]

Figure: Unequal vortices not collinear with the observation.

Three Filters of Two Vortex Flow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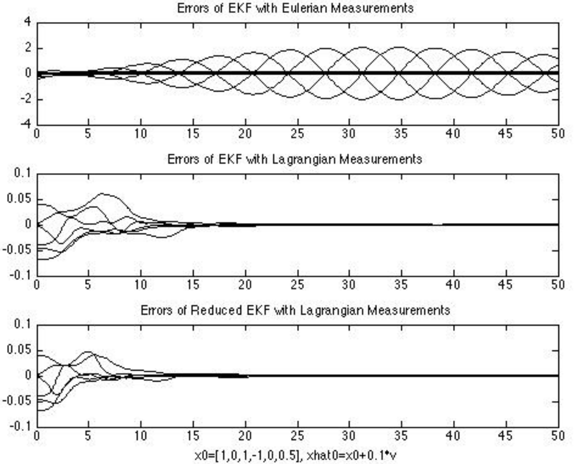

Figure: Unequal vortices collinear with the observation. Initial estimation error in the null space of the SORC one forms.

Three Filters of Two Vortex Flow

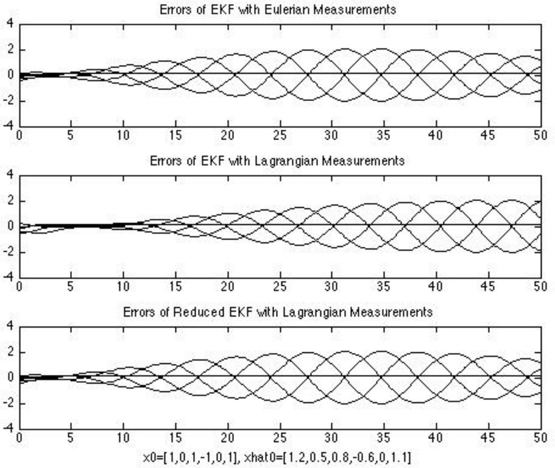

Figure: Two equal vortices symmetric with respect to the observation.

Conclusion

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How do we target observation locations to maximize observability?