Data Assimilation and Detection in Multi-Sensor & Multi-Scale Environments

N. Sri Namchchivaya
Andrea Barreiro, Shanshan Liu, Kristjan Onu, Jun Park
Pete Sauer, and Richard Sowers

Department of Aerospace Engineering
University of Illinois at Urbana-Champaign

February 5, 2008
Banff International Research Station

NSF CMMI 04-01412, DMS 05-04581 and CNS-0540216

Dynamic Data Driven Power Systems (**DDDPS**)

The main objectives:

1. Combine computational models with sensor data to predict the dynamics of large-scale evolving systems.

Figure: Electric Power Grid of United States

2. Improve the ability to dynamically steer large-scale complex systems and the measurement processes.

Signal & Observation Processes

We consider a nonlinear $\mathbb{R}^d$-valued *signal process X^ε*

$$dX_t^\varepsilon = b_\varepsilon(X_t^\varepsilon)dt + \sigma_\varepsilon(X_t^\varepsilon)dV_t, \quad X_0^\varepsilon = \xi$$

and an $\mathbb{R}^n$-valued *observation process Y^ε* given by the SDE

$$dY_t^\varepsilon = h_\varepsilon(X_t^\varepsilon)dt + dB_t, \quad Y_0^\varepsilon = 0$$

where V and B are independent Wiener processes and ξ is a random initial condition which is independent of V and B.

Signal process X^ε is composed of slow and fast variables such that the *generator* $\mathscr{L}_\varepsilon$ of the Markov process X^ε is of the form

$$\mathscr{L}_\varepsilon\varphi = \frac{1}{\varepsilon}\mathscr{L}_F\varphi + \mathscr{L}_S\varphi,$$

for all $\varepsilon \in (0,1)$ (denotes the scale separation) and all $\varphi \in C^\infty(\mathbb{R}^d)$, where $\mathscr{L}_F$ and $\mathscr{L}_S$ represent generators of fast and slow variables.

What is Filtering

Estimate the signal X_t^ε at time t based on the information in the observation Y^ε up to time t;

$$\mathscr{Y}_t^\varepsilon \overset{\text{def}}{=} \sigma\{Y_s^\varepsilon : 0 \le s \le t\}.$$

More precisely for each $t \ge 0$, we want to compute the *conditional law* of X_t^ε given $\mathscr{Y}_t^\varepsilon$

$$\pi_t^\varepsilon(A) \overset{\text{def}}{=} \mathbb{P}\{X_t^\varepsilon \in A | \mathscr{Y}_t^\varepsilon\}$$

for all $A \in \mathscr{B}(\mathbb{R}^d)$. We then have (with respect to Lebesgue measure on $\mathbb{R}^d$)

$$\pi_t^\varepsilon(A) = \int_{x\in A} p^\varepsilon(t,x)dx = \frac{\int_{x\in A} u^\varepsilon(t,x)dx}{\int_{x\in\mathbb{R}^d} u^\varepsilon(t,x)dx},$$

where we can directly define

$$p^\varepsilon(t,x) \overset{\text{def}}{=} \frac{u^\varepsilon(t,x)}{\int_{x'\in\mathbb{R}^d} u^\varepsilon(t,x')dx'}.$$

Filtering Equations

One can construct a linear recursive filtering equation for the un-normalized density $u^{\varepsilon}(t,x)$ via a stochastic PDE (Zakai equation):

$$du^{\varepsilon}(t,x) = \mathscr{L}_{\varepsilon}^{*} u^{\varepsilon}(t,x)dt + u^{\varepsilon}(t,x)h_{\varepsilon}(x)dY_t^{\varepsilon}, \qquad u^{\varepsilon}(0,\cdot) = p_{\xi}$$

where $\mathscr{L}_{\varepsilon}^{*}$ is the adjoint operator of the $\mathscr{L}_{\varepsilon}$ and the initial condition ξ has density p_{ξ}.

The main difficulty:

- The numerical solutions of such stochastic PDEs get prohibitively expensive as the state dimension increases.

Goals:

Find a data-driven low-order model to extract useful information

- for more accurate and long-term assesment of the system
- for real-time detection of extreme events

Procedures: combine two ingredients, namely, stochastic dimensional reduction and nonlinear filtering.

1. Dimensional Reduction of Nonlinear Filters
   - Reduction of Signal Processes: $X_t^\varepsilon \Rightarrow \overline{X_t}$
   - Reduction of Nonlinear Filters: $\pi^\varepsilon \Rightarrow \bar{\pi}$
2. Approximate Filters via Particle Methods: Construction of *lower-dimensional particle filters*.
   - $\lim_{N\to\infty}(\bar{\Pi}_N(t), \phi) = \bar{\pi}_t(\phi)$
3. Application
   - Vortex Dynamics

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Original signal process

Our *starting point* is the signal process $\{X_t^\varepsilon\}$ on $\mathbb{R}^n$ whose generator is

$$\mathscr{L}_\varepsilon \varphi = \frac{1}{\varepsilon}\mathscr{L}_F \varphi + \mathscr{L}_S \varphi.$$

We will construct this Markov process in a canonical way in $C([0,\infty);\mathbb{R}^n)$, via the martingale problem.

Martingale Problem (Strook and Varadhan)

Define the event space $\Omega \stackrel{\text{def}}{=} C([0,\infty);\mathbb{R}^n)$ with coordinate functions $X_t(\omega) \stackrel{\text{def}}{=} \omega(t)$ for all $t \geq 0$ and all $\omega \in \Omega$. Let $f \in C^2(\mathbb{R}^n)$.

Then

$$M_t^{f,\varepsilon} \stackrel{\text{def}}{=} f(x) - \int_0^t (\mathscr{L}^\varepsilon f)(X_u)du$$

is a martingale with respect to the filtration $\{\mathscr{F}_t;\ t \geq 0\}$ under the probability measure $\mathbb{P}^\varepsilon$ (i.e. $\mathbb{E}^\varepsilon[M_t^{f,\varepsilon}|\mathscr{F}_s] = M_s^{f,\varepsilon}$ for all $0 \leq s \leq t$) .

Reduced State Space

Let the unperturbed ($\varepsilon = 0$) flow ($\mathfrak{z}_t$) generate an equivalence relation on the original state space $\mathbb{R}^n$.

We say that x and y in $\mathbb{R}^n$ are equivalent, i.e., $x \sim y$, if $\mathfrak{z}_t(x) = y$ for some $t \in \mathbb{R}$. If $x \in \bar{\mathbf{S}} \subset \mathbb{R}^n$, we let $[x] \stackrel{\text{def}}{=} \{y \in \bar{\mathbf{S}} : \ y \sim x\}$ be the equivalence class of x and we define $\pi(x) \stackrel{\text{def}}{=} [x]$. Define

$$\mathfrak{M} \stackrel{\text{def}}{=} \bar{\mathbf{S}}/\sim .$$

The dimension of $\mathfrak{M}$ is much smaller than n. The discontinuities are intrinsic to the reduced state space $\mathfrak{M}$ (due to “quotioning”).

Example: Liquid Sloshing Motion

Consider just two wave modes and their Hamiltonian. Here $\mathfrak{M}$ looks like a bunch of intersecting planes.

$\mathfrak{M}$ consists of three planes, $\mathcal{I}_i$, each of which has H, I as the local coordinates and are joined at the vertex, $z = \mathcal{O}$, to form an ”arrowhead”.

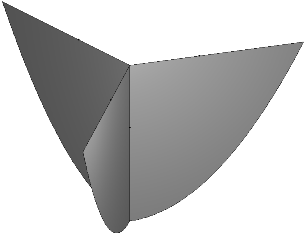


Figure: Reduced space.

Reduced Markov Process

Define now a process in the reduced state space $\mathfrak{M}$,

$$Z_t \stackrel{\text{def}}{=} [X_t], \qquad t \geq 0.$$

The main difficulty is the presence of *discontinuities* in the statistics of the Markov processes; mathematically, this is equivalent to a study of *boundary layer* problems for stochastic processes.

We prove that the $\mathbb{P}^\varepsilon$-law of $\{[X_t];\ t \geq 0\}$ tends to a $\mathfrak{M}$-valued Markov process with an identifiable generator.

Our main result will be that the $\mathbb{P}^\varepsilon$'s tend to the unique solution $\mathbb{P}^\dagger$ of the martingale problem with generator $\mathscr{L}^\dagger$, with domain $\mathscr{D}^\dagger$ and with initial condition $\delta_{[x]}$.

Arnold,Namachchivaya & Schenk [1996], Freidlin & Weber [1998], Namachchivaya & Sowers [2001, 2002], Namachchivaya & Van Roessel [2003] Freidlin & Wentzell [2004]

Example: Liquid Sloshing Motion

If we define the slowly-varying quantity $Z : \mathbb{R}^4 \to \mathbb{R}^2$ by

$$Z(x) \stackrel{\text{def}}{=} (H(x), I(x)), \qquad x \in \mathbb{R}^4 \tag{1}$$

Our goal is to show that as ε tends to zero, the dynamics of $Z_t^\varepsilon = Z(X_t^\varepsilon)$ tends to a lower-dimensional Markov process and to identify the generator $\mathscr{L}^\dagger$ of the limiting law, $\mathbb{P}^\dagger$. For $f \in \mathscr{D}^\dagger$, we define

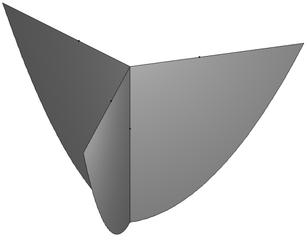
$$(\mathscr{L}^\dagger f)(z) \stackrel{\text{def}}{=} \sum_{j=1}^{2} \mathfrak{b}_i(z)\frac{\partial f}{\partial z_j}(z) + \frac{1}{2}\sum_{j,k=1}^{2} \mathfrak{a}_{jk}(z)\frac{\partial^2 f}{\partial z_j \partial z_j}(z) \tag{2}$$

for all $z = (z_1, z_2) \in \mathcal{I}$. At the line where the planes of the arrowhead meet, *gluing conditions* define the behavior of the process.

The limiting domain $\mathscr{D}^\dagger$ for the graph valued process is

$$\mathscr{D}^\dagger_{\mathfrak{M}} = \Big\{ f \in C(\mathfrak{M}) \cap C^2(\cup_{i=1}^3 \mathcal{I}_i) : \lim_{z\searrow(H(\mathfrak{c}_i),I(\mathfrak{c}_i))} (\mathscr{L}_i f_i)(h) \quad \text{exists} \quad \forall i,$$

$$\lim_{z_2\nearrow I^*} (\mathscr{L}_i f_i)(z)=0 \quad \forall i, \quad \underbrace{\sum_{i=1}^{3}\{\pm\}\sum_{j=1}^{2}\Big\{\sum_{k=1}^{2} \mathring{\mathfrak{a}}^{i}_{jk}(z)\frac{\partial f_i}{\partial z_k}(z)\Big\}\cdot\nu_j\Big|_{z=\mathcal{O}}=0}_{\textit{glueing conditions}}\Big\}$$



Dimensional Reduction of Nonlinear Filters

The main objectives:

1. To show that if the signal process $\{X_t^\varepsilon\}$ of a singularly perturbed stochastic differential equation converges to coarse-grained process $\{\overline{X}_t\}$ in a weak sense, then the nonlinear filter process $\{\pi_t^\varepsilon\}$ will converge to process $\{\bar{\pi}_t\}$. The reduced nonlinear filter process $\{\bar{\pi}_t\}$ is governed by a lower dimensional Zakai equation.
2. For the reduced nonlinear model an appropriate form of particle filter can be a viable and useful scheme — numerically solve the lower dimensional Zakai equation by an appropriate form of particle filter.

Question: How does scaling interact with filtering?

Consider a *plant* given by an $\mathbb{R}^2$-valued stochastic differential equation

$$\begin{aligned} d\Theta_t^\varepsilon &= \varepsilon^{-1/2} dW_t \qquad t \geq 0, \quad \Theta_0^\varepsilon = \theta_\circ, \\ dZ_t^\varepsilon &= \sigma(\Theta_t^\varepsilon) dV_t \qquad t \geq 0, \quad Z_0^\varepsilon = z_\circ. \end{aligned}$$

We interpret Z^ε as a slow axial variable and Θ^ε as a fast angle variable.

The *observation* process will be given by

$$Y_t^\varepsilon = \int_{s=0}^t h(\Theta_s^\varepsilon) ds + B_t. \qquad t \geq 0$$

Our goal is to study the *conditional law* of the plant on the basis of the observations. We expect that to be able to average out the effects of the fast variable Θ^ε.

For each $t \geq 0$, define $\mathscr{Y}_t^\varepsilon \overset{\text{def}}{=} \sigma\{Y_s^\varepsilon;\, 0 \leq s \leq t\}$ and define the $C([0,t];\mathbb{R})$-valued random variable $Y_{[0,t]}^\varepsilon$ as $Y_s^\varepsilon(\omega)$ for all $\omega \in \Omega$ and $s \in [0,t]$. For each $t \geq 0$, there is a measurable map

$$\pi_t^\varepsilon : C([0,t];\mathbb{R}) \to \mathscr{P}(\mathbb{R})$$

such that for each $t \geq 0$ and and $A \in \mathscr{B}(\mathbb{R})$,

$$\pi_t^\varepsilon(A;\, Y_{[0,t]}^\varepsilon) = \mathbb{P}[Z_t^\varepsilon \in A | \mathscr{Y}_t^\varepsilon].$$

We note that the π_t^ε's give the conditional law of only the *slow* component Z^ε of the plant.

Average out the effects of the fast variable Θ^{ε}. Define

$$\bar{\sigma} \stackrel{\text{def}}{=} \left\{ \int_{\theta=0}^{1} \sigma^2(\theta) d\theta \right\}^{1/2} \quad \text{and} \quad \bar{h} \stackrel{\text{def}}{=} \int_{\theta=0}^{1} h(\theta) d\theta.$$

Then the reduced plant and observation processes are

$$d\overline{Z}_t = \bar{\sigma} dV_t, \quad \overline{Z}_0 = z_\circ, \quad t \geq 0$$
$$\overline{Y}_t \stackrel{\text{def}}{=} \int_{s=0}^{t} \bar{h}\, ds + B_t = \bar{h} t + B_t, \quad t \geq 0$$

where the averaged observation process is independent of the plant. Since $\overline{Z}_t$ is Gaussian, for $t > 0$, define

$$\bar{p}_t(x) \stackrel{\text{def}}{=} \int_{x' \in \mathbb{R}} (2\pi \bar{\sigma}^2 t)^{-1/2} \exp\left[-\frac{(x - x')^2}{2\bar{\sigma}^2 t} \right] p_0(dx')$$

Set

$$\bar{\pi}_t(A) \overset{\text{def}}{=} \int_{x \in A} \bar{p}_t(x)dx$$

for all $A \in \mathscr{B}(\mathbb{R})$. We then have that

$$\mathbb{P}\left\{\overline{X}_t \in A | \overline{\mathscr{Y}}_t\right\} = \bar{\pi}_t(A)$$

for all $t \geq 0$ and $A \in \mathscr{B}(\mathbb{R})$, where $\overline{\mathscr{Y}}_t \overset{\text{def}}{=} \sigma\{\overline{Y}_s;\ 0 \leq s \leq t\}$.

Our claim is , if you are only interested in finding the distribution of the slow variables, you might as well use the averaged dynamics, that is,

Theorem (Park, Namachchivaya, Sowers (2007))

Let $d_{\mathscr{P}(\mathbb{R})}$ is the standard Prohorov metric on $\mathscr{P}(\mathbb{R})$. For each $t > 0$,

$$\lim_{\varepsilon \searrow 0} \mathbb{E}\left[d_{\mathscr{P}(\mathbb{R})}(\pi_t^\varepsilon(\cdot, Y_{[0,t]}^\varepsilon), \bar{\pi}_t(\cdot))\right] = 0.$$

Since the averaged observation process is independent of the plant, the averaged filter can throw away the observations.

An important quantity in studying limits of filtering problems is

$$\exp\left[-\int_{s=0}^{t} h(\Theta_s^\varepsilon)dB_s - \frac{1}{2}\int_{s=0}^{t} h^2(\Theta_s^\varepsilon)ds\right]. \tag{3}$$

If this quantity should converges as $\varepsilon \searrow 0$ to a random variable with mean 1 (i.e., a Radon-Nikodym derivative), then the filtering problems should converge. In our case, it does *not*.

If h depends on Z^ε the result is more interesting. Then the averaged observation process depends on the plant and we show that $\pi_t^\varepsilon(\cdot, Y_{[0,t]}^\varepsilon)$ is close to $\bar{\pi}_t(\cdot, Y_{[0,t]}^\varepsilon)$.

Detection of Vortices

Consider a stochastic two-vortex model that approximates the evolution of vorticity with viscosity (Marchioro and Pulvirenti [1985], Ide, Kuznetsov & Jones [2002]))

$$dx_t^1 = -\frac{\Gamma_2}{2\pi}\frac{(x_t^2 - x_t^4)}{r^2}dt + \sqrt{2\nu}dW_t^1, \; dx_t^2 = \frac{\Gamma_1}{2\pi}\frac{(x_t^1 - x_t^3)}{r^2}dt + \sqrt{2\nu}dW_t^2$$

$$dx_t^3 = -\frac{\Gamma_2}{2\pi}\frac{(x_t^4 - x_t^2)}{r^2}dt + \sqrt{2\nu}dW_t^3, \; dx_t^4 = \frac{\Gamma_1}{2\pi}\frac{(x_t^3 - x_t^1)}{r^2}dt + \sqrt{2\nu}dW_t^4,$$

where (x_1, x_2) and (x_3, x_4) represent the position coordinates of the first and second vortices respectively.

Introduce *relative* and "center of mass" coordinates as

$$x^r = x^3 - x^1, \; y^r = x^4 - x^2; \quad x^c = \frac{\Gamma_1 x^1 + \Gamma_2 x^3}{\Gamma_1 + \Gamma_2}, \; y^c = \frac{\Gamma_1 x^2 + \Gamma_2 x^4}{\Gamma_1 + \Gamma_2}.$$

Define

$$z \stackrel{\text{def}}{=} \{x^r, y^r, x^c, y^c\}, \quad \tau \stackrel{\text{def}}{=} \Gamma_1 + \Gamma_2, \quad \text{and} \quad \kappa_i \stackrel{\text{def}}{=} \frac{\Gamma_i}{\Gamma_1 + \Gamma_2}$$

Then the generator of the Markov process is given by

$$
\begin{aligned}
(\mathscr{L} f) =& \frac{\tau}{2\pi}\,\frac{1}{z_1^2+z_2^2}\left(-z_2\frac{\partial f}{\partial z_1}+z_1\frac{\partial f}{\partial z_2}\right)+2\,\nu\left(\frac{\partial^2 f}{\partial z_1^2}(z)+\frac{\partial^2 f}{\partial z_2^2}(z)\right)\\
&+\nu\left(\kappa_1^2+\kappa_2^2\right)\left(\frac{\partial^2 f}{\partial z_3^2}(z)+\frac{\partial^2 f}{\partial z_4^2}(z)\right)\\
&+2\nu\left(\kappa_2-\kappa_1\right)\left(\frac{\partial^2 f}{\partial z_1\partial z_3}(z)+\frac{\partial^2 f}{\partial z_2\partial z_4}(z)\right)
\end{aligned}
$$

for $f \in C^2(\mathbb{R}^4)$. The probability density is governed by the forward Kolmogorov equation

$$
\begin{aligned}
\frac{\partial}{\partial t}P(z,t|\mathscr{Y}_{t_k}) =& \mathscr{L}^* P(z,t|\mathscr{Y}_{t_k}), \quad t_k < t < t_{k+1},\\
\text{with} \quad \lim_{t\to t_k} P(z,t|\mathscr{Y}_{t_k}) =& P(z,t_k|\mathscr{Y}_{t_k}).
\end{aligned}
$$

where $\mathscr{L}^*$ is the adjoint operator.

The probability density for two-vortices with equal strengths ($\kappa_2 = \kappa_1$) is

$$P(x^r, y^r, x^c, y^c, t) = p_r(x^r, y^r, t) p_c(x^c, y^c, t) \tag{4}$$

where

$$p_r(x^r, y^r, t) = \frac{1}{4\pi\nu t} \int\int d\xi d\eta \, e^{-(|\bar{x}^r|^2 + |\bar{\xi}|^2)/(4\nu t)} \times \left[\sum_{p \in Z} e^{ip \tan^{-1}(y^r/x^r) - ip \tan^{-1}(\eta/\xi)} I_{\mu_p}\left(\frac{|\bar{x}^r||\bar{\xi}|}{2\nu t}\right) \right] p_r(\xi, \eta, 0),$$

$$p_c(x^c, y^c, t) = \frac{1}{\pi\nu t} e^{-((x^c)^2 + (y^c)^2)/(\nu t)},$$

$\bar{x}^r \stackrel{\text{def}}{=} (x^r, y^r)$, $\bar{\xi} \stackrel{\text{def}}{=} (\xi, \eta)$, $I_m(z)$ is the modified Bessel function of the first kind with order m and argument z and $\mu_p^2 = p^2 + ip/\nu$, and the root should be chosen so that $Re(\mu_p) \leq 0$.

Numerical Results: Evolution of a pair of vortices

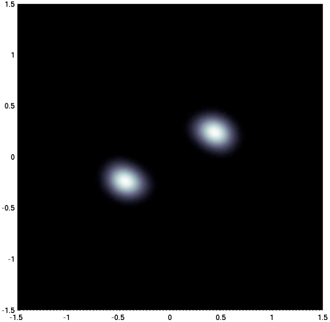


Figure: The left figure shows the superimposed distributions of (x^1, x^2) and (x^3, x^4) at $t = 1$. The right figure shows $t = 5$.

Discrete Observations: Tracer Advection

The observations are defined by the m tracers and are taken at discrete time instants t_k.

$$y_k^i = h_i(z_k, y_{k-1}) + v_k^i, \qquad z_k = z_{t_k}, \quad y_k^i = y_{t_k}^i, \qquad i = 1 \dots 2m \quad (5)$$

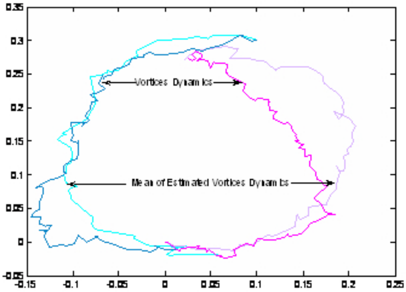
Between observations, the conditional **pdf** $p(z, t|\mathscr{F}_t^y)$ is given by the explicit solution (4) at any time $t > t_k$.

At time $t = t_{k+1}$, we get more information from the observation y_{k+1}, which is used to *update* this conditional **pdf** at $t = t_{k+1}$.

Since $v_k \sim N(0, R_k)$, by Bayes' rule, we can explicitly write

$$p\left(y_k \middle| z, y\right) = \frac{1}{(2\pi)^{\frac{m}{2}} |R_k|^{\frac{1}{2}}} \exp\{-\frac{1}{2}\left(y_k - h(z, y, t_k)\right)^T R_k^{-1} \left(y_k - h(z, y, t_k)\right)\}$$

Numerical Results: Particle Filters



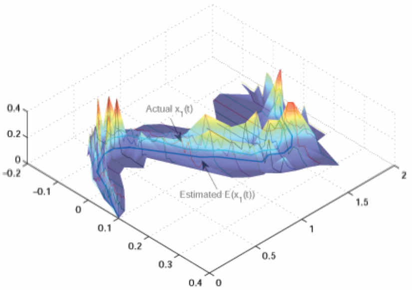


Figure: The left figure shows the mean value of estimated position of the vortices by tracking the single tracer. The right figure shows the conditioned pdf of the position

Data Fusion: Particle Filters with Multiple Sensors

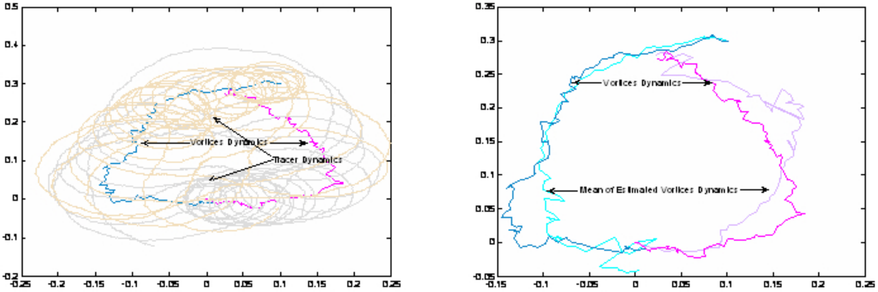


Figure: The left figure shows the vortex-tracer dynamics.The right figure shows that with two or more tracers, the extraction results can be improved.