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Lagrangian data assimilation for point vortex systems†

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Abstract. A new method for directly assimilating Lagrangian tracer observations for flow state estimation is presented. It is developed in the context of point vortex systems. With tracer advection equations augmenting the point vortex model, the correlations between the vortex and tracer positions allow one to use the observed tracer positions to update the non-observed vortex positions. The method works efficiently when the observations are accurate and frequent enough. Low-quality data and large intervals between observations can lead to divergence of the scheme. Nonlinear effects, responsible for the failure of the extended Kalman filter, are triggered by the exponential rate of separation of tracer trajectories in the neighbourhoods of the saddle points of the velocity field.

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1. Introduction

Estimating the true state of the flow from observations is a most challenging task. Most importantly, there has been no mathematical framework for incorporating Lagrangian observations, such as time series of tracer positions, into the computational model of the flow. In this study, we present a new method for assimilating Lagrangian data based on a sequential approach following the well-known (extended) Kalman filter [1].

We describe the flow state at time t by an $N_{\rm F}$-dimensional vector $\boldsymbol{x}_{\rm F}(t)$ where subscript 'F' stands for 'flow'. The evolution of the true state $\boldsymbol{x}_{\rm F}^{\rm t}(t)$, with superscript 't' for 'true' [2], is governed by the $N_{\rm F}$-dimensional nonlinear stochastic differential equation

$$\mathrm{d}\boldsymbol{x}_{\rm F}^{\rm t} = M_{\rm F}(\boldsymbol{x}_{\rm F}^{\rm t}, t)\,\mathrm{d}t + \mathrm{d}\boldsymbol{\eta}_{\rm F}^{\rm t}. \tag{1}$$

The additive stochastic noise $\boldsymbol{\eta}$ represents the unknown external forcing, as well as unresolved sub-grid-scale processes and other errors of the deterministic model given by $M_{\rm F}$. In the sequential data assimilation framework, $\boldsymbol{\eta}$ is often assumed to be a Wiener process with zero mean, $E[\boldsymbol{\eta}] = 0$, and covariance $E[\boldsymbol{\eta}(\boldsymbol{\eta})^{\rm T}] \equiv \boldsymbol{Q}$, denoted by $\boldsymbol{\eta} \sim N(\boldsymbol{0}, \boldsymbol{Q}^{\rm t})$. Here $E[\cdot]$ and $(\cdot)^{\rm T}$ stand for the expectation and transpose operators.

The true flow state $\boldsymbol{x}_{\rm F}^{\rm t}(t)$ is unknown. Taking an expectation of (1) gives the $N_{\rm F}$-dimensional deterministic equation. If an appropriate set of initial conditions is available, then it leads to a forecast $\boldsymbol{x}_{\rm F}^{\rm f}(t)$ of $\boldsymbol{x}_{\rm F}^{\rm t}(t)$. While $\boldsymbol{x}_{\rm F}^{\rm f}(t)$ can be simulated over time, observations $\boldsymbol{y}^{\rm o}$ become available in a discrete time sequence. At time t_k, the L_k-dimensional $\boldsymbol{y}_k^{\rm o}$ may offer a(n incomplete) description of the state.

The goal of data assimilation is to obtain the optimal estimate of $\boldsymbol{x}_{\rm F}^{\rm t}(t)$ by combining the two independent pieces of information, i.e., model forecast $\boldsymbol{x}_{\rm F}^{\rm f}(t)$ and observation $\boldsymbol{y}_k^{\rm o}$. In the standard formulation of the extended Kalman filtering, this is possible if and only if $\boldsymbol{y}_k^{\rm o}$ depends on $\boldsymbol{x}_{\rm F}^{\rm t}(t_k)$ explicitly.

Difficulties, therefore, arise in assimilating the Lagrangian data into the models. For example, let us consider a Lagrangian tracer. Given the velocity from $\boldsymbol{x}_{\rm F}^{\rm t}(t)$, a true Lagrangian tracer $\boldsymbol{x}_{\rm D}^{\rm t}(t)$ moves with the local flow subject to some stochastic noise:

$$\mathrm{d}\boldsymbol{x}_{\rm D}^{\rm t} = M_{\rm D}(\boldsymbol{x}_{\rm F}^{\rm t}, \boldsymbol{x}_{\rm D}^{\rm t}, t)\,\mathrm{d}t + \mathrm{d}\boldsymbol{\eta}_{\rm D}^{\rm t}, \tag{2}$$

where subscript 'D' stands for Lagrangian drifter data and the nonlinear dynamic model $M_{\rm D}$ depends on both $\boldsymbol{x}_{\rm F}$ and $\boldsymbol{x}_{\rm D}$. At t_k, the instantaneous tracer position $\boldsymbol{x}_{\rm D}^{\rm t}(t)$ alone offers absolutely no information concerning the current state $\boldsymbol{x}_{\rm F}^{\rm t}(t_k)$, although it is a direct consequence of $\boldsymbol{x}_{\rm F}^{\rm t}(t)$ over time as in (2).

2. Methodology

In order to overcome the difficulties, we introduce a new data assimilation scheme based on the extended Kalman filter. Our innovation is to define the state vector $\boldsymbol{x}$ as a combination of not only the flow states but also the Lagrangian tracers [3]:

$$\boldsymbol{x} \equiv \begin{pmatrix} \boldsymbol{x}_{\rm F} \\ \boldsymbol{x}_{\rm D} \end{pmatrix}. \tag{3a}$$

Therefore, the error covariance matrix defined by

$$\boldsymbol{P} \equiv E[(\boldsymbol{x} - \boldsymbol{x}^{\rm t})(\boldsymbol{x} - \boldsymbol{x}^{\rm t})^{\rm T}] = E\begin{bmatrix} (\boldsymbol{x}_{\rm F} - \boldsymbol{x}_{\rm F}^{\rm t})(\boldsymbol{x}_{\rm F} - \boldsymbol{x}_{\rm F}^{\rm t})^{\rm T} & (\boldsymbol{x}_{\rm F} - \boldsymbol{x}_{\rm F}^{\rm t})(\boldsymbol{x}_{\rm D} - \boldsymbol{x}_{\rm D}^{\rm t})^{\rm T} \\ (\boldsymbol{x}_{\rm D} - \boldsymbol{x}_{\rm D}^{\rm t})(\boldsymbol{x}_{\rm F} - \boldsymbol{x}_{\rm F}^{\rm t})^{\rm T} & (\boldsymbol{x}_{\rm D} - \boldsymbol{x}_{\rm D}^{\rm t})(\boldsymbol{x}_{\rm D} - \boldsymbol{x}_{\rm D}^{\rm t})^{\rm T} \end{bmatrix} \tag{3b}$$

includes the error correlation between $\boldsymbol{x}_{\rm F}$ and $\boldsymbol{x}_{\rm D}$. Moreover, an observation

$$\boldsymbol{y}_k^{\rm o} = h_k(\boldsymbol{x}^{\rm t}(t_k)) + \boldsymbol{\epsilon}_k^{\rm t} \tag{4}$$

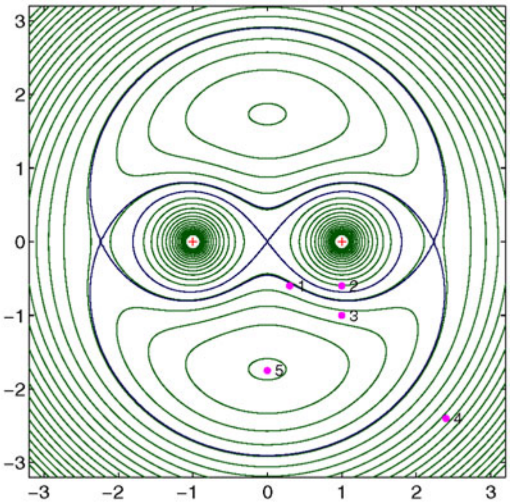

Figure 1. The stream-function field in the co-rotating frame with the vortices at (−1, 0) and (1, 0); the rotation period is $T = (4\pi)^2$. The numbered circles indicate the initial positions of the two $\boldsymbol{x}_{\mathrm{D}}(0)$ (see also table 2).

becomes dependent on the state vector $\boldsymbol{x}^{\mathrm{t}}(t_k)$, subject to stochastic error $\boldsymbol{\epsilon}_k^{\mathrm{t}} \sim N(0, \boldsymbol{R}_k^{\mathrm{t}})$. Whether or not $\boldsymbol{y}_k^{\mathrm{o}}$ is limited to Lagrangian tracers, information from $\boldsymbol{y}_k^{\mathrm{o}}$ propagates through $\boldsymbol{x}$ via the correlation terms as in (3*b*). Thus, this new scheme allows us to assimilate the Lagrangian observation directly as a passive tracer and yet also update the flow state in the model.

The extended Kalman filter attempts to determine the optimal estimate $\boldsymbol{x}^{\mathrm{a}}$ of $\boldsymbol{x}^{\mathrm{t}}$ by minimizing the global measure of the error defined by tr $\boldsymbol{P}^{\mathrm{a}}$ (tr = trace), as $\boldsymbol{y}_k^{\mathrm{o}}$ becomes available sequentially in time. One assimilation cycle over $[t_{k-1}, t_k]$ consists of the following two steps.

Step 1

Dynamic forecast from time t_{k-1} to t_k. Given an initial condition $\boldsymbol{x}^{\mathrm{f}}(t_{k-1})$ and $\boldsymbol{P}^{\mathrm{f}}(t_{k-1})$, a forecast can be obtained deterministically from

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}^{\mathrm{f}} = M(\boldsymbol{x}^{\mathrm{f}}, t) \tag{5a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P}^{\mathrm{f}} = \boldsymbol{M}\boldsymbol{P}^{\mathrm{f}} + \boldsymbol{P}^{\mathrm{f}}(\boldsymbol{M})^{\mathrm{T}} + \boldsymbol{Q}^{\mathrm{f}}. \tag{5b}$$

The error covariance is advanced using the tangent linear model:

$$\boldsymbol{M} \equiv J[M(\boldsymbol{x}^{\mathrm{f}}, t), \boldsymbol{x}^{\mathrm{f}}] \tag{5c}$$

evaluated along $\boldsymbol{x}^{\mathrm{f}}(t)$, where J denotes the Jacobian matrix. Because $\boldsymbol{\eta}^{\mathrm{t}}(t)$ is unknown, $\boldsymbol{Q}^{\mathrm{f}}(t)$ represents the best guess for $\boldsymbol{Q}^{\mathrm{t}}(t)$.

Step 2

Update by probabilistic analysis at t_k. The optimal-analysis $\boldsymbol{x}_k^{\mathrm{a}}$ and $\boldsymbol{P}_k^{\mathrm{a}}$ are obtained by combining the forecast and observation:

$$\boldsymbol{x}_k^{\mathrm{a}} = \boldsymbol{x}^{\mathrm{f}}(t_k) + \boldsymbol{K}_k(\boldsymbol{y}_k^{\mathrm{o}} - h_k(\boldsymbol{x}^{\mathrm{f}}(t_k))), \tag{6a}$$

$$\boldsymbol{P}_k^{\mathrm{a}} = (\boldsymbol{I} - \boldsymbol{K}_k\boldsymbol{H}_k)\boldsymbol{P}^{\mathrm{f}}(t_k), \tag{6b}$$

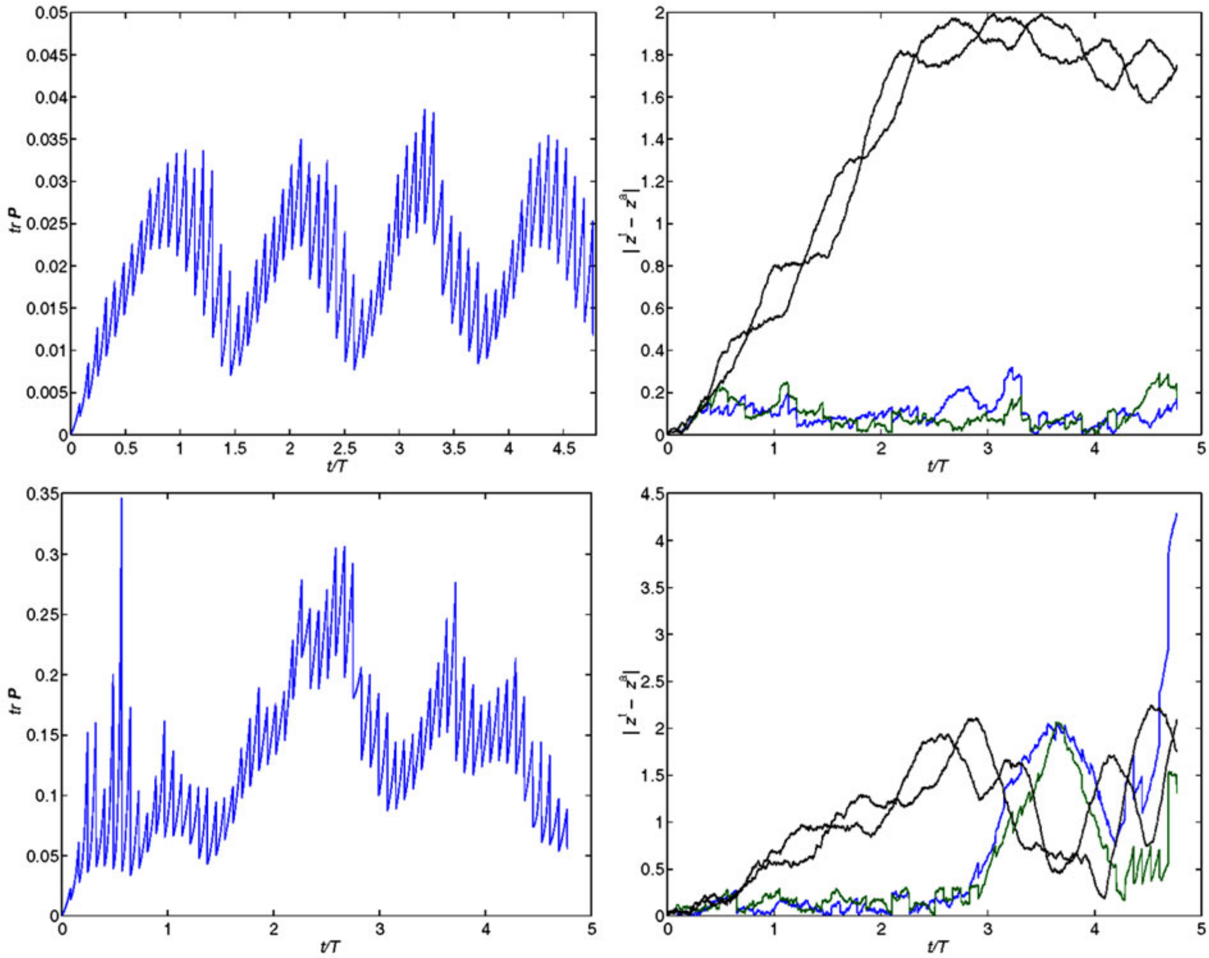

Figure 2. The dependence on (σ, ρ) for $\Delta t = 1.0$ with $\boldsymbol{x}_{\mathrm{D}}^{\mathrm{f}}(0) = (0.3, -0.6)$ (number 1 in figure 1): $(\sigma, \rho) = (0.02, 0.02)$ (top panels) and $(\sigma, \rho) = (0.05, 0.05)$ (bottom panels), for $\mathrm{tr}\,\boldsymbol{P}$ (left column) and actual distance δ (right column; the black lines correspond to the case without Lagrangian data assimilation). In the bottom panels with a larger noise level, the scheme fails after $t > 3T$.

where the superscript 'a' stands for 'analysis' and $\boldsymbol{I}$ is the identity matrix. The weight $\boldsymbol{K}_k$ is called the 'Kalman gain' and given as a ratio of the forecast and observational errors:

$$\boldsymbol{K}_k = \boldsymbol{P}^{\mathrm{f}}(\boldsymbol{H}_k)^{\mathrm{T}}(\boldsymbol{H}_k\boldsymbol{P}^{\mathrm{f}}\boldsymbol{P}^{\mathrm{f}}(\boldsymbol{H}_k)^{\mathrm{T}} + \boldsymbol{R}_k^{\mathrm{o}})^{-1}, \tag{7a}$$

$$\boldsymbol{H}_k = J[h_k(\boldsymbol{x}^{\mathrm{f}}), \boldsymbol{x}^{\mathrm{f}}], \tag{7b}$$

where $\boldsymbol{R}^{\mathrm{o}}$ is the best guess for the true observation error covariance $\boldsymbol{R}^{\mathrm{t}}$.

The next assimilation cycle over $[t_k, t_{k+1}]$ starts with the initial condition $\boldsymbol{x}^{\mathrm{f}}(t_k) = \boldsymbol{x}_k^{\mathrm{a}}$ and $\boldsymbol{P}^{\mathrm{f}}(t_k) = \boldsymbol{P}_k^{\mathrm{a}}$ in step 1.

3. Numerical experiments

The new scheme is tested on point vortex flows as a first step towards realistic Lagrangian data assimilation for the flows dominated by coherent vortex structures [4]. The state $\boldsymbol{x}_{\mathrm{F}}(t)$ is described by the position of N_v point vortices ($N_{\mathrm{F}} = 2N_v$), and observation $\boldsymbol{y}_k^{\mathrm{o}}$ has N_t

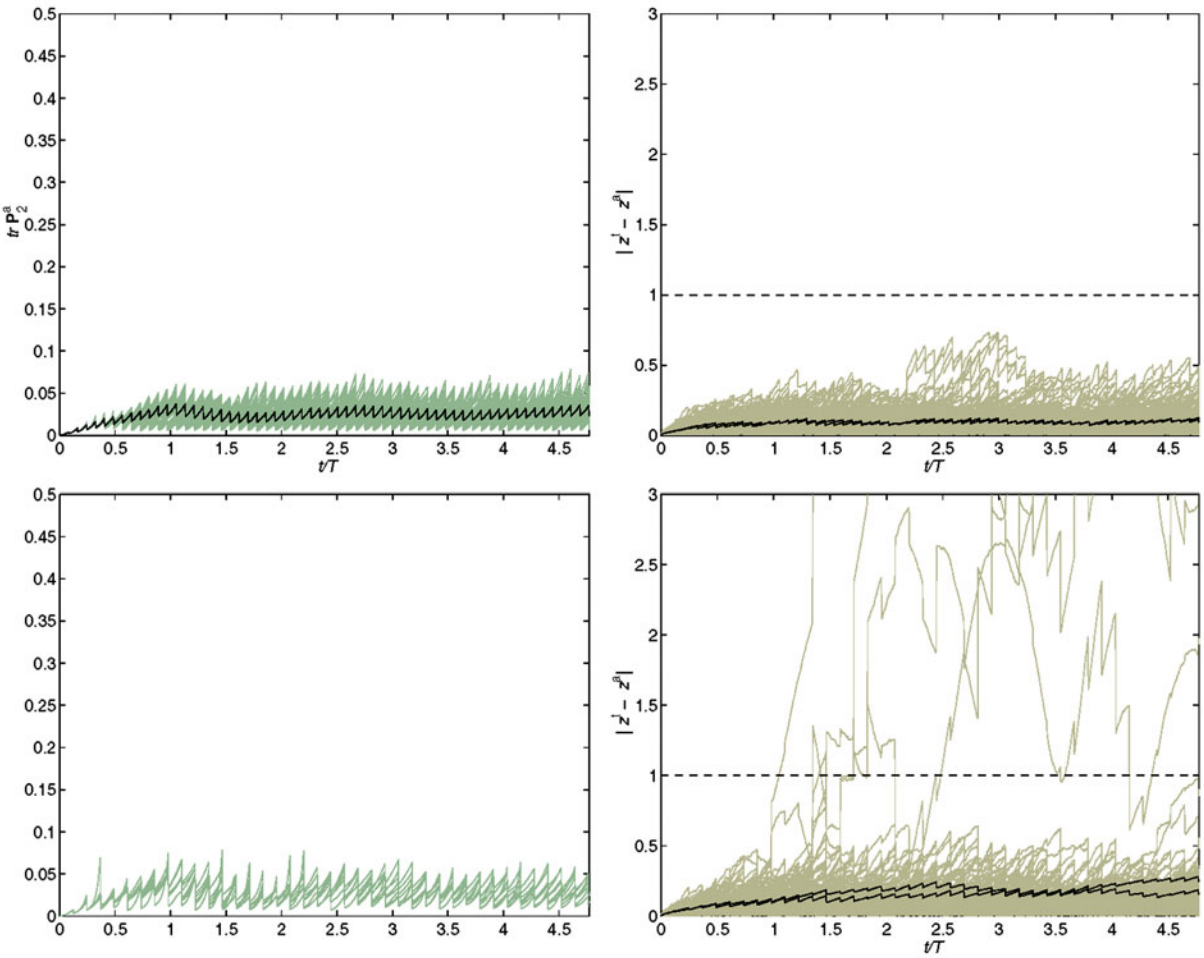

Figure 3. The dependence on Δt for $(\sigma, \rho) = (0.02, 0.02)$ with $\boldsymbol{x}_{\mathrm{D}}^{\mathrm{f}}(0) = (0.3, -0.6)$ (number 1 in figure 1); 100 realizations for the thick curve on average: $\Delta t = 1.0$ (top panels) and $\Delta t = 1.5$ (bottom panels), for $\mathrm{tr}\,\boldsymbol{P}$ (left column) and actual distance δ (right column). In the top panels, all realizations show successful assimilation. In the bottom panels with a larger Δt, two out of 100 realizations fail (see also table 1).

tracer positions with $h_k(\boldsymbol{x}(t_k)) = \boldsymbol{x}_{\mathrm{D}}(t_k)$ at any k $(L_k = 2N_t)$. We conduct the identical twin experiments where the true state $\boldsymbol{x}_{\mathrm{F}}^{\mathrm{t}}(t)$ and observations $\boldsymbol{y}_k^{\mathrm{o}}$ are obtained by numerical simulation of the model subject to stochastic noises and errors as in (1), (2), and (4).

In this controlled setting, the distance between assimilated and true state vectors

$$\delta^{\mathrm{a,f}}(t) = |\boldsymbol{x}^{\mathrm{a,f}}(t) - \boldsymbol{x}^{\mathrm{t}}(t)| \tag{8}$$

can be used to check whether $\boldsymbol{x}^{\mathrm{a,f}}(t)$ appropriately represents $\boldsymbol{x}^{\mathrm{t}}(t)$. An assimilation experiment is said to be a 'failure' if $\delta^{\mathrm{a,f}}(t)$ surpasses a certain threshold δ_f. In addition, the failure percentage p_f based on the δ_f criteria can be computed over a number of realizations subject to stochastic noise and errors. Self-evaluation of the overall performance for the extended Kalman filter is given by $\mathrm{tr}\,\boldsymbol{P}^{\mathrm{a,f}}(t)$. If an assimilation experiment is successful, then $\mathrm{tr}\,\boldsymbol{P}^{\mathrm{a,f}}$ remains small for all time. Moreover, $\mathrm{tr}\,\boldsymbol{P}^{\mathrm{a,f}} \approx (\delta^{\mathrm{a,f}}(t))^2$ should hold by definitions (3b) and (8).

Table 1. The dependence on Δt for $(\sigma,\rho)=(0.02,0.02)$ where $\langle\delta\rangle$ is the average of $\delta^{\mathrm{f,a}}(t)$ (figure 3).

Δt	0.75	1.0	1.25	1.5	1.75	2.0
p_f	0	0	2	2	8	18
$\langle\delta\rangle$	0.09	0.10	0.12	0.19	0.35	0.51

Table 2. The dependence of $\boldsymbol{x}^{\mathrm{f}}_{\mathrm{D}}(0)$; the vortex identity number of $\boldsymbol{x}^{\mathrm{f}}_{\mathrm{D}}(0)$ corresponds to that shown in figure 1.

No	1	2	3	4	5
$\boldsymbol{x}^{\mathrm{f}}_{\mathrm{D}}(0)$	$(0.3,-0.6)$	$(1,-0.6)$	$(1,-1)$	$(2.4,-2.4)$	$(0,-1.75)$
p_f	0.5	55.5	0.5	15	0
$\langle\delta\rangle$	0.12	1.90	0.11	0.292	0.11

In order to reveal the basic mechanism of Lagrangian data assimilation, we focus on a simple case with $(N_v,N_t)=(2,1)$. Without stochastic noise, a pair of vortices rotate around each other along a circle at the constant angular velocity $2\pi/T$ where T is the period of the vortex rotation. Such flow dynamics is completely integrable. However, the induced velocity field is highly nonlinear, as shown by the streamlines in the co-rotating frame with the vortices (figure 1). For simplicity, the dynamic noise and observational error are described by

$$\boldsymbol{Q}^{\mathrm{f}}=2\sigma^2\boldsymbol{I},\qquad \boldsymbol{R}^{\mathrm{o}}=2\rho^2\boldsymbol{I}. \tag{9}$$

We set the failure threshold $\delta_f=1$. Hence, parameters of such identical twin experiments are: noise level (σ,ρ), assimilation cycle interval $\Delta t\equiv t_k-t_{k-1}$, and initial tracer position $\boldsymbol{x}^{\mathrm{f}}_{\mathrm{D}}(0)$.

The dependence on (σ,ρ) is shown in figure 2. Our new scheme for the Lagrangian data assimilation works successfully for lower noise levels. As the noise level rises, it may fail after some time. The dependence on Δt is shown in figure 3 and table 1. Not surprisingly, Lagrangian data assimilation works better for smaller Δt.

In contrast to the dependence on (σ,ρ) and Δt, the dependence on $\boldsymbol{x}^{\mathrm{f}}_{\mathrm{D}}(0)$ is subtle due to the nonlinearity in the velocity field (figure 1). Table 2 summarizes the results of ensemble experiments. Initial condition number 2 fails most because it is too close to the vortex and hence the nonlinear effect misleads the covariance evolution which is obtained by the tangent linear model (5c). Number 4 fails because it is too far from the vortices and hence the error correlation between $\boldsymbol{x}_{\mathrm{F}}$ and $\boldsymbol{x}_{\mathrm{D}}$ in (3b) is too weak for updating $\boldsymbol{x}_{\mathrm{F}}$ with $\boldsymbol{x}_{\mathrm{D}}$ properly. Number 1 is the initial condition corresponding to the experiments for the dependence on (σ,ρ) and Δt (figures 1–3, and table 1). Strong error correlation between $\boldsymbol{x}_{\mathrm{F}}$ and $\boldsymbol{x}_{\mathrm{D}}$ tends to give good performance. It occasionally fails when $\boldsymbol{x}^{\mathrm{f}}_{\mathrm{F}}(t)$ and/or $\boldsymbol{x}^{\mathrm{t}}_{\mathrm{F}}(t)$ approach one of the separatrices too closely: $\boldsymbol{x}^{\mathrm{f}}_{\mathrm{D}}(t)$ and $\boldsymbol{x}^{\mathrm{t}}_{\mathrm{D}}(t)$ may end up across the separatrix and hence eventually draw apart as they pass the saddle. While nonlinearity in the velocity field leads to the presence of more than one saddle and separatrix, separation at the saddle is a linear effect. Number 3 performs about the same as number 1; it has weaker error correlation between $\boldsymbol{x}_{\mathrm{F}}$ and $\boldsymbol{x}_{\mathrm{D}}$ but also little danger of going near the separatrix.

4. Concluding remarks

We presented a new data assimilation scheme that allows us to assimilate Lagrangian data directly into the model. Our scheme is based on the extended Kalman filter. Representing the state vector as a combination of flow states and Lagrangian tracer positions, information from Lagrangian observations can propagate into the flow state through the error correlation between them. The scheme is successfully applied to point vortex flows. Linear and nonlinear effects are shown to influence the performance of the Lagrangian data assimilation scheme.

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