A Complementary Theory of Light Scattering by Homogeneous Spheres

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ABSTRACT

A theory of the scattering of electromagnetic waves by homogeneous spheres, the so-called Mie theory, is presented in a unique and coherent manner in this paper. We begin with Maxwell's equations, from which the vector wave equations are derived and solved by means of the two orthogonal solutions to the scalar wave equation. The transverse incident electric field is mapped in spherical coordinates and expanded in known mathematical functions satisfying the scalar wave equation. Determination of the unknown coefficients in the scattered and internal fields is achieved by matching the electromagnetic boundary conditions on the surface of a sphere. Far-field solutions for the electric field are then given in terms of the scattering functions. Transformation of the electric field to the reference plane containing incident and scattered waves is carried out. Extinction parameters and the phase matrix are derived from the electric field perpendicular and parallel to the reference plane. On the basis of the independent-scattering assumption, the theory is extended to cases involving a sample of homogeneous spheres.

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1. INTRODUCTION

The elegance and intricacy of the theory of light scattering by homogeneous spheres have been greatly hindered by a number of gaps in its development. Mathematical and physical deductions on light scattering by spheres in the past have not been presented in such a way that beginners in atmospheric radiation and optics can follow them, and that researchers in various areas can assimilate the development logically for applications to theoretical and practical problems on planetary atmospheres. It is the

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purpose of this paper to provide a coherent development on the theory of light scattering by homogeneous spheres.

Kerker [1] has described comprehensively the history of the solution for scattering by a homogeneous sphere. Credit for the theory has gone to Mie [2], as evidenced by the common term "Mie scattering". Debye [3], however, made an equal contribution by introducing the potential named after him to solve Maxwell's equations. Earlier contributors to the theory of scattering by a sphere have been cited by Kerker, and I shall not repeat his excellent review of the subject.

A number of textbooks have treated elegantly the subject of light scattering by a homogeneous sphere. Among them, Stratton [4] was the first to present this subject in an organized manner. His approach, however, was very general, and one has to put together pieces from various sections in order to have a picture of the scattering theory as a whole. Born and Wolf [5] provided a more tractable approach, utilizing the Debye potential, up to the derivation of the scattered electric and magnetic fields. van de Hulst [6] followed much of Stratton's work, and simplified the presentation of the theory. In his early chapters he provided physical foundations for the determination of extinction parameters and the phase function. Kerker [1] presented a brief analysis of the scattering theory, employing Born and Wolf's approach. More detailed discussions on the scattering theory were also given by Saxon [7].

The Mie theory is essentially concerned with scattering of electromagnetic waves by *one* homogeneous sphere. Applications of such a theory to scattering processes in planetary atmospheres containing particles of various sizes can be accomplished only if independent scattering is assumed. It is on the assumption of particle independence that Mie scattering theory is applied to most atmospheric problems. For the computational aspects of light scattering by a sample of homogeneous spheres, Deirmendjian [8] gave comprehensive discussions and presented useful numerical results in terms of tables. In a similar vein, Hansen and Travis [9], among many others, provided practical information on light scattering by the homogeneous and spherical cloud and aerosol particles that are typical in planetary atmospheres.

In this paper, we start from Maxwell's equations in Sec. 2 to obtain the governing equations for the electric and magnetic field vectors. Section 3 gives the derivation of the solution of the vector wave equation in spherical coordinates. In Sec. 4 the formal solution of the scattering problem (i.e., the determination of scattering coefficients) is presented. In Sec. 5, I discuss the far field solution and show how the electric field is transferred to a plane containing scattered and incident waves as originally described by van de Hulst [6]. Derivations of extinction parameters including extinction and scattering cross-sections (Sec. 6) and phase matrix (Sec. 7) then follow.

Lastly, applications of scattering theory to a sample of large homogeneous spheres with a distribution function are discussed in Sec. 8. Problems of Raman (or fluorescent) scattering are not considered in this study.

2. MAXWELL'S EQUATIONS

The state of excitation which is established in space by electric charges constitutes the electromagnetic field. It is represented by two vectors, **E** and **B**, called the electric field and magnetic induction, respectively. It is necessary to introduce a second set of vectors, i.e., the electric current density j, the electric displacement **D** and the magnetic vector **H**, to describe effects of the electromagnetic field on material objects. At every point in whose neighborhood the physical properties of the medium are continuous, the space and time derivatives of these five vectors are related by Maxwell's equations,

$$\nabla\times\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}+\frac{4\pi}{c}\mathbf{j}, \tag{2.1}$$

$$\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \tag{2.2}$$

$$\nabla\cdot\mathbf{D}=4\pi\rho, \tag{2.3}$$

$$\nabla\cdot\mathbf{B}=0, \tag{2.4}$$

where t denotes time, c velocity of light and ρ the density of charge. Equation (2.3) may be regarded as a defining equation for the electric charge density ρ, and Eq. (2.4) may be interpreted as saying that there exist no free magnetic poles. Here the so-called Gaussian system of units is used.

From Eq. (2.1), since $\nabla\cdot\nabla\times\mathbf{H}=0$, we have

$$\nabla\cdot\mathbf{j}=-\frac{1}{4\pi}\nabla\cdot\frac{\partial\mathbf{D}}{\partial t}, \tag{2.5}$$

and from Eq. (2.3),

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{j}=0. \tag{2.6}$$

This is the equation of continuity of the electromagnetic field.

To allow a unique determination of the field vectors from a given distribution of current and charges, these equations must be supplemented by relations describing the behavior of substances under the influence of the field. These relations are given by

$$\mathbf{j}=\sigma\mathbf{E}, \tag{2.7}$$

$$\mathbf{D}=\varepsilon\mathbf{E}, \tag{2.8}$$

$$\mathbf{B}=\mu\mathbf{H}, \tag{2.9}$$

where σ is the specific conductivity, ε the dielectric constant (or permittivity) and μ the magnetic permeability.

We shall now confine our attention to the field where there are no charges ($\rho=0$) or currents ($\mathbf{j}=0$) and where the medium is homogeneous, so that ε and μ are constants. Thus, Maxwell's equations become

$$\nabla\times\mathbf{H}=\frac{\varepsilon}{c}\frac{\partial\mathbf{E}}{\partial t}, \tag{2.10}$$

$$\nabla\times\mathbf{E}=\frac{-\mu}{c}\frac{\partial\mathbf{H}}{\partial t}, \tag{2.11}$$

$$\nabla\cdot\mathbf{E}=0, \tag{2.12}$$

$$\nabla\cdot\mathbf{H}=0. \tag{2.13}$$

Equations (2.10)–(2.13) are to be used to derive the electromagnetic wave equation. Note here that Eqs. (2.12) and (2.13) are immediate consequences of Eqs. (2.10) and (2.11).

3. THE ELECTROMAGNETIC WAVE EQUATION AND ITS SOLUTION

We consider a plane electromagnetic wave in a periodic field with a circular frequency ω, so we make the replacement

$$\mathbf{E}\rightarrow\mathbf{E}e^{i\omega t}, \tag{3.1a}$$

$$\mathbf{H}\rightarrow\mathbf{H}e^{i\omega t}. \tag{3.1b}$$

Then (2.10) and (2.11) yield

$$\nabla\times\mathbf{H}=ikm^2\mathbf{E}, \tag{3.2}$$

$$\nabla\times\mathbf{E}=-ik\mathbf{H}, \tag{3.3}$$

with

$$k=\frac{\omega}{c}=\frac{2\pi}{\lambda}, \tag{3.4}$$

$$m^2=\varepsilon, \tag{3.5}$$

where k is the wavenumber (the propagation constant in vacuum), λ is the wavelength in vacuum, m is the complex refractive index of the medium at the frequency ω and the permeability $\mu\approx 1$ for air.

We now perform the curl operation on Eq. (3.3):

$$\nabla\times\nabla\times\mathbf{E}=-ik\nabla\times\mathbf{H}. \tag{3.6}$$

By noting that $\nabla\cdot\nabla\times\mathbf{E}=0$, we have

$$\nabla^2\mathbf{E}=-k^2m^2\mathbf{E}. \tag{3.7}$$

Similarly,

$$\nabla^2\mathbf{H}=-k^2m^2\mathbf{H}. \tag{3.8}$$

Equations (3.7) and (3.8) indicate that the electric vector and magnetic induction in a homogeneous medium satisfy the following vector wave equation:

$$\nabla^2\mathbf{A}+k^2m^2\mathbf{A}=0. \tag{3.9}$$

Now, if ψ satisfies the scalar wave equation

$$\nabla^2\psi+k^2m^2\psi=0, \tag{3.10}$$

then the vectors $\mathbf{M}_\psi$ and $\mathbf{N}_\psi$ in spherical coordinates (r,θ,ϕ) defined by

$$\begin{aligned}\mathbf{M}_\psi&=\nabla\times(\mathbf{a}_r\psi)\\&=\mathbf{a}_\theta\frac{1}{r\sin\theta}\frac{\partial(r\psi)}{\partial\phi}-\mathbf{a}_\phi\frac{1}{r}\frac{\partial(r\psi)}{\partial\theta},\end{aligned} \tag{3.11}$$

$$mk\mathbf{N}_\psi = \nabla \times \mathbf{M}_\psi$$

$$= \mathbf{a}_r\left[\frac{\partial^2(r\psi)}{\partial r^2} + m^2k^2r\psi\right] + \mathbf{a}_\theta \frac{1}{r}\frac{\partial^2(r\psi)}{\partial r\,\partial\phi}$$

$$+ \mathbf{a}_\phi \frac{1}{r\sin\theta}\frac{\partial^2(r\psi)}{\partial r\,\partial\phi} \tag{3.12}$$

satisfy the vector wave equation defined in Eq. (3.9) subject to Eq. (3.10). $\mathbf{a}_r$, $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$ are unit vectors in spherical coordinates.

Assume that u and v are two orthogonal solutions of the scalar wave equation defined in Eq. (3.10). Then the electric and magnetic field vectors expressed by

$$\mathbf{E} = \mathbf{M}_v + i\mathbf{N}_u, \tag{3.13}$$

$$\mathbf{H} = m(-\mathbf{M}_u + i\mathbf{N}_v) \tag{3.14}$$

satisfy Eqs. (3.2) and (3.3). Employing Eqs. (3.11) and (3.12), **E** and **H** can be written explicitly as follows:

$$\mathbf{E} = \mathbf{a}_r \frac{i}{mk}\left[\frac{\partial^2(ru)}{\partial r^2} + m^2k^2ru\right]$$

$$+ \mathbf{a}_\theta\left[\frac{1}{r\sin\theta}\frac{\partial(rv)}{\partial\phi} + \frac{i}{mkr}\frac{\partial^2(ru)}{\partial r\,\partial\theta}\right]$$

$$+ \mathbf{a}_\phi\left[-\frac{1}{r}\frac{\partial(rv)}{\partial\theta} + \frac{1}{mkr\sin\theta}\frac{\partial^2(ru)}{\partial r\,\partial\phi}\right], \tag{3.15}$$

$$\mathbf{H} = \mathbf{a}_r \frac{i}{k}\left[\frac{\partial^2(rv)}{\partial r^2} + m^2k^2rv\right]$$

$$+ \mathbf{a}_\theta\left[-\frac{m}{r\sin\theta}\frac{\partial(ru)}{\partial\phi} + \frac{i}{kr}\frac{\partial^2(rv)}{\partial r\,\partial\theta}\right]$$

$$+ \mathbf{a}_\phi\left[\frac{m}{r}\frac{\partial(ru)}{\partial\theta} + \frac{i}{kr\sin\theta}\frac{\partial^2(rv)}{\partial r\,\partial\phi}\right]. \tag{3.16}$$

The scalar wave equation defined in Eq. (3.10) is given in spherical coordinates by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}+k^2m^2\psi=0. \quad (3.17)$$

This equation is separable by letting

$$\psi(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi). \quad (3.18)$$

Upon substituting Eq. (3.18) into Eq. (3.17) and dividing the entire equation by $\psi(r,\theta,\phi)$, we obtain

$$\frac{1}{r^2}\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{1}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}+k^2m^2=0.$$

$$(3.19)$$

If Eq. (3.19) is multiplied by $r^2\sin^2\theta$, then we have

$$\left[\sin^2\theta\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\sin\theta\frac{1}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+k^2m^2r^2\sin^2\theta\right]+\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}=0.$$

$$(3.20)$$

Since the first three terms in this equation contain the variables r and θ only, the only way Eq. (3.20) can be valid is if

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=\text{constant}=-l^2, \quad (3.21)$$

where we set the constant equal to $-l^2$ with l an integer for mathematical convenience. It is also clear that

$$\sin^2\theta\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\sin\theta\frac{1}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+k^2m^2r^2\sin^2\theta-l^2=0. \quad (3.22)$$

Upon dividing Eq. (3.22) by $\sin^2\theta$, we obtain

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+k^2m^2r^2+\frac{1}{\sin\theta}\frac{1}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)-\frac{l^2}{\sin^2\theta}=0. \tag{3.23}$$

Evidently, in order to satisfy Eq. (3.23),

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+k^2m^2r^2=\text{constant}=n(n+1), \tag{3.24}$$

$$\frac{1}{\sin\theta}\frac{1}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{l^2}{\sin^2\theta}=\text{constant}=-n(n+1), \tag{3.25}$$

where n is an integer. The selection of the constant here is also for mathematical convenience. Rearranging Eqs. (3.21), (3.24) and (3.25), we have

$$\frac{d^2(rR)}{dr^2}+\left[k^2m^2-\frac{n(n+1)}{r^2}\right]rR=0, \tag{3.26}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[n(n+1)-\frac{l^2}{\sin^2\theta}\right]\Theta=0, \tag{3.27}$$

$$\frac{d^2\Phi}{d\phi^2}+l^2\Phi=0. \tag{3.28}$$

For Eq. (3.28), the single-valued solution is

$$\Phi=a_l\cos l\phi+b_l\sin l\phi, \tag{3.29}$$

where a_l and b_l are arbitrary constants.

Equation (3.27) is the well-known equation for spherical harmonics. We introduce a new variable $\xi=\cos\theta$, so that

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\Theta}{d\xi}\right]+\left[n(n+1)-\frac{l^2}{1-\xi^2}\right]\Theta=0. \tag{3.30}$$

The solutions of Eq. (3.30) are the associated Legendre polynomials (spherical harmonics of the first kind):

$$\Theta=P_n^l(\xi)=P_n^l(\cos\theta). \tag{3.31}$$

To integrate the remaining equation (3.26), we set

$$kmr=\rho, \qquad R=\frac{1}{\sqrt{\rho}}Z(\rho). \tag{3.32}$$

Thus,

$$\frac{d^2Z}{d\rho^2}+\frac{1}{\rho}\frac{dZ}{d\rho}+\left[1-\frac{\left(n+\frac{1}{2}\right)^2}{\rho^2}\right]Z=0. \tag{3.33}$$

The solution of this equation is the general cylindrical function of order $n+\frac{1}{2}$ and is given by

$$Z=Z_{n+1/2}(\rho). \tag{3.34}$$

The solution of Eq. (3.26) is then

$$R=\frac{1}{\sqrt{kmr}}Z_{n+1/2}(kmr). \tag{3.35}$$

The elementary wave function at all points on the surface of a sphere is therefore given by

$$\psi(r,\theta,\phi)=\frac{1}{\sqrt{kmr}}Z_{n+1/2}(kmr)P_n^l(\cos\theta)(a_l\cos l\phi+b_l\sin l\phi). \tag{3.36}$$

Each cylindrical function in Eq. (3.35) may be expressed as a linear combination of two cylindrical functions of standard type, e.g., the Bessel functions $J_{n+1/2}(\rho)$ and the Neumann functions $N_{n+1/2}(\rho)$. We define

$$\psi_n(\rho)=\sqrt{\frac{\pi\rho}{2}}\,J_{n+1/2}(\rho), \tag{3.37}$$

$$\chi_n(\rho)=-\sqrt{\frac{\pi\rho}{2}}\,N_{n+1/2}(\rho). \tag{3.38}$$

The functions ψ_n are regular in every finite domain of the ρ-plane including the origin, whereas the functions χ_n have singularities at the origin $\rho=0$ where they become infinite. Hence, we may use ψ_n but not χ_n to represent the wave inside the sphere.

Equation (3.35) can be rewritten as

$$rR = c_n \psi_n(kmr) + d_n \chi_n(kmr), \tag{3.39}$$

where c_n and d_n are arbitrary constants. Equation (3.39) represents the general solution of Eq. (3.26).

The general solution of the scalar wave equation can then be expressed as

$$\psi(r,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{l=-n}^{n} P_n^l(\cos\theta)\left[c_n\psi_n(kmr) + d_n\chi_n(kmr)\right](a_l\cos l\phi + b_l \sin l\phi). \tag{3.40}$$

From Eqs. (3.15) and (3.16), the electric and magnetic field vectors of electromagnetic waves can be subsequently derived.

We note here that when $c_n = 1$, $d_n = i$,

$$\psi_n(\rho) + i\chi_n(\rho) = \sqrt{\frac{\pi\rho}{2}}\, H_{n+1/2}^{(2)}(\rho) = \zeta_n(\rho), \tag{3.41}$$

where $H_{n+1/2}^{(2)}$ is the half-integral-order Hankel function of the second kind. It has the property of vanishing at infinity in the complex plane and is suitable for the representation of the scattered wave.

4. FORMAL SCATTERING SOLUTION

With the vector wave equation solved, we may now discuss the scattering of a plane wave by a homogeneous sphere. We assume for simplicity that outside the medium is vacuum ($m = 1$), that the material of the sphere has an index of refraction m, and that the incident radiation is linearly polarized. We select the origin of a rectangular system of coordinates at the center of the sphere, with the positive z-axis along the direction of propagation of the incident wave. If the amplitude of the incident wave is normalized to unity, the incident electric and magnetic field vectors are

$$\mathbf{E}^i = \mathbf{a}_x e^{-ikz}, \tag{4.1a}$$

$$\mathbf{H}^i = \mathbf{a}_y e^{-ikz}, \tag{4.1b}$$

where $\mathbf{a}_x$ and $\mathbf{a}_y$ are unit vectors along the x- and y-axes.

As shown in Fig. 1, the components of any vector, say **a**, are transformed from the Cartesian system to the spherical polar coordinates r, θ and ϕ defined by

$$x = r\sin\theta\cos\phi, \tag{4.2a}$$

$$y = r\sin\theta\sin\phi, \tag{4.2b}$$

$$z = r\cos\theta, \tag{4.2c}$$

according to the geometrical relationship

$$\mathbf{a}_r = \mathbf{a}_x \sin\theta\cos\phi + \mathbf{a}_y \sin\theta\sin\phi + \mathbf{a}_z \cos\theta, \tag{4.3a}$$

$$\mathbf{a}_\theta = \mathbf{a}_x \cos\theta\cos\phi + \mathbf{a}_y \cos\theta\sin\phi - \mathbf{a}_z \sin\theta, \tag{4.3b}$$

$$\mathbf{a}_\phi = -\mathbf{a}_x \sin\phi + \mathbf{a}_y \cos\phi, \tag{4.3c}$$

where $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$ are unit vectors along x, y, z, respectively, and $\mathbf{a}_r$, $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$ are unit vectors in spherical coordinates.

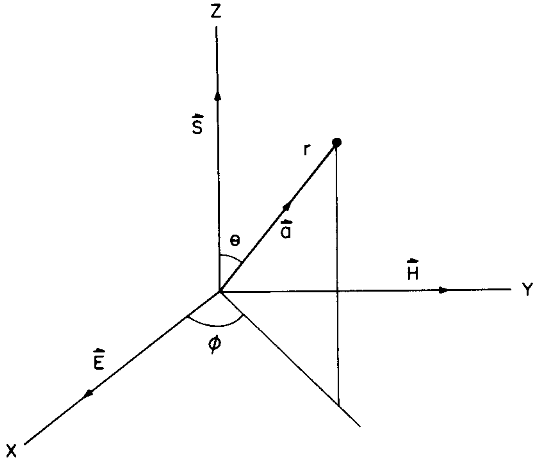

FIG. 1. Transformation of rectangular to spherical coordinates. S is the Poynting vector, and a is an arbitary unit vector.

Thus, the electric and magnetic field vectors of the incident wave are

$$E_r^i = e^{-ikr\cos\theta}\sin\theta\cos\phi, \tag{4.4a}$$

$$E_\theta^i = e^{-ikr\cos\theta}\cos\theta\cos\phi, \tag{4.4b}$$

$$E_\phi^i = -e^{-ikr\cos\theta}\sin\phi, \tag{4.4c}$$

$$H_r^i = e^{-ikr\cos\theta}\sin\theta\sin\phi, \tag{4.5a}$$

$$H_\theta^i = e^{-ikr\cos\theta}\cos\theta\sin\phi, \tag{4.5b}$$

$$H_\phi^i = e^{-ikr\cos\theta}\cos\phi. \tag{4.5c}$$

We note that the first factor on the left-hand side of this equation may be expressed in the following differentiable series of Legendre polynomials:

$$e^{-ikr\cos\theta} = \sum_{n=0}^{\infty}(-i)^n(2n+1)\frac{\psi_n(kr)}{kr}P_n(\cos\theta), \tag{4.6}$$

where ψ_n is defined in Eq. (3.37a). We also have the identities

$$e^{-ikr\cos\theta}\sin\theta = \frac{1}{ikr}\frac{\partial}{\partial\theta}(e^{-ikr\cos\theta}), \tag{4.7}$$

$$\frac{\partial}{\partial\theta}P_n(\cos\theta) = -P_n^1(\cos\theta), \qquad P_0^1(\cos\theta)=0. \tag{4.8}$$

Equation (4.8) relates the Lengendre polynomial P_n to the associated Legendre polynomial P_n^1.

To determine the potentials u and v, only one of the components in (3.15) is needed. The first of them is $(m=1)$

$$E_r^i = e^{-ikr\cos\theta}\sin\theta\cos\phi = \frac{i}{k}\left[\frac{\partial^2(ru^i)}{\partial r^2} + k^2ru^i\right]. \tag{4.9}$$

In view of Eqs. (4.6), (4.7) and (4.8), we have

$$e^{-ikr\cos\theta}\sin\theta\cos\phi = \frac{1}{(kr)^2}\sum_{n=1}^{\infty}(-i)^{n-1}(2n+1)\psi_n(kr)P_n^1(\cos\theta)\cos\phi. \tag{4.10}$$

Accordingly we take a trial solution in Eq. (4.9) in a series with a similar form:

$$ru^i = \frac{1}{k}\sum_{n=1}^{\infty} \alpha_n \psi_n(kr) P_n^1(\cos\theta)\cos\phi. \tag{4.11}$$

Upon substituting Eqs. (4.10) and (4.11) into Eq. (4.9) and comparing coefficients, we obtain

$$\alpha_n\left[k^2\psi_n(kr) + \frac{\partial^2\psi_n(kr)}{\partial r^2}\right] = (-i)^{n-2}(2n+1)\frac{\psi_n(kr)}{r^2}. \tag{4.12}$$

From Eq. (3.39), since the $\chi_n(kr)$ become infinite at the origin (through which the incident wave must pass), we may let $c_n = 1$, $d_n = 0$. Then

$$\psi_n(kr) = rR \tag{4.13}$$

is a solution of Eq. (3.26) (with $m = 1$):

$$\frac{d^2\psi_n}{dr^2} + \left[k^2 - \frac{\alpha}{r^2}\right]\psi_n = 0, \tag{4.14}$$

provided that $\alpha = n(n+1)$. Comparing Eq. (4.14) with (4.12), we see that

$$\alpha_n = (-i)^{n-2}\frac{2n+1}{n(n+1)}. \tag{4.15}$$

With a similar procedure, v^i can be derived from Eq. (3.16). Thus, for the incident wave outside the sphere we have

$$ru^i = \frac{1}{k}\sum_{n=1}^{\infty}(-i)^{n-2}\frac{2n+1}{n(n+1)}\psi_n(kr)P_n^1(\cos\theta)\cos\phi, \tag{4.16a}$$

$$rv^i = \frac{1}{k}\sum_{n=1}^{\infty}(-i)^{n-2}\frac{2n+1}{n(n+1)}\psi_n(kr)P_n^1(\cos\theta)\sin\phi. \tag{4.16b}$$

In order to match u^i and v^i with those of internal and scattered waves, whose potentials have already been derived in Eq. (3.40), the latter must be expressed in a series of similar form but with arbitrary coefficients. For internal waves, the function $\chi_n(kmr)$ becomes infinite at the origin, so that only the function $\psi_n(kmr)$ may be used. Thus, for internal waves we have

$$ru^t = \frac{1}{mk}\sum_{n=1}^{\infty}(-i)^{n-2}\frac{2n+1}{n(n+1)}c_n\psi_n(kmr)P_n^1(\cos\theta)\cos\phi, \tag{4.17a}$$

$$rv^t = \frac{1}{mk}\sum_{n=1}^{\infty}(-i)^{n-2}\frac{2n+1}{n(n+1)}d_n\psi_n(kmr)P_n^1(\cos\theta)\sin\phi. \tag{4.17b}$$

For scattered waves, the solutions must vanish at infinity, and the Hankel functions expressed in Eq. (3.41) will impart precisely this property. Thus for scattered waves we have

$$ru^s = -\frac{1}{k}\sum_{n=1}^{\infty}(-i)^{n-2}\frac{2n+1}{n(n+1)}a_n\zeta_n(kr)P_n^1(\cos\theta)\cos\phi, \tag{4.18a}$$

$$rv^s = -\frac{1}{k}\sum_{n=1}^{\infty}(-i)^{n-2}\frac{2n+1}{n(n+1)}b_n\zeta_n(kr)P_n^1(\cos\theta)\sin\phi. \tag{4.18b}$$

The coefficients a_n, b_n, c_n and d_n have to be determined from the boundary conditions at the surface of the sphere. Note here that these coefficients differ from those in Eqs. (3.40). The boundary conditions are that the tangential components of **E** and **H** are continuous across the spherical surface $r=a$. So we have

$$\left.\begin{aligned} E_\theta^i + E_\theta^s &= E_\theta^t \\ E_\phi^i + E_\phi^s &= E_\phi^t \\ H_\theta^i + H_\theta^s &= H_\theta^t \\ H_\phi^i + H_\phi^s &= H_\phi^t \end{aligned}\right\} \quad \text{at} \quad r=a. \qquad \left\{\begin{aligned} &(4.19\text{a}) \\ &(4.19\text{b}) \\ &(4.19\text{c}) \\ &(4.19\text{d}) \end{aligned}\right.$$

In view of Eqs. (4.16), (4.17), (4.18) and Eqs. (3.15), (3.16), it is evident that apart from common factors and differentiations with respect to θ, ϕ, which are the same for the wave inside and outside the sphere, the field components E_θ and E_ϕ both contain the expressions v and $\partial(ru)/m\,\partial r$. It is also clear that components H_θ and H_ϕ contain mu and $\partial(rv)/\partial r$. Equation (4.19) implies that these four expressions have to be continuous at $r=a$.

Consequently,

$$\frac{\partial}{\partial r}\left[r(u^i+u^s)\right]=\frac{1}{m}\frac{\partial}{\partial r}(ru^t), \tag{4.20a}$$

$$\frac{\partial}{\partial r}\left[r(v^i+v^s)\right]=\frac{\partial}{\partial r}(rv^t), \tag{4.20b}$$

$$u^i+u^s=mu^t, \tag{4.20c}$$

$$v^i+v^s=v^t. \tag{4.20d}$$

From these equations, it is now apparent that

$$m\left[\psi_n'(ka)-a_n\zeta_n'(ka)\right]=c_n\psi_n'(kma), \tag{4.21a}$$

$$\left[\psi_n'(ka)-b_n\zeta_n'(ka)\right]=d_n\psi_n'(kma), \tag{4.21b}$$

$$\left[\psi_n(ka)-a_n\zeta_n(ka)\right]=c_n\psi_n(kma), \tag{4.21c}$$

$$m\left[\psi_n(ka)-b_n\zeta_n(ka)\right]=d_n\psi_n(kma), \tag{4.21d}$$

where the prime denotes differentiation with respect to the argument. Upon eliminating c_n and d_n, we obtain the coefficients for scattered waves:

$$a_n=\frac{\psi_n'(y)\psi_n(x)-m\psi_n(y)\psi_n'(x)}{\psi_n'(y)\zeta_n(x)-m\psi_n(y)\zeta_n'(x)}, \tag{4.22a}$$

$$b_n=\frac{m\psi_n'(y)\psi_n(x)-\psi_n(y)\psi_n'(x)}{m\psi_n'(y)\zeta_n(x)-\psi_n(y)\zeta_n'(x)}, \tag{4.22b}$$

where $x=ka$ and $y=mx$. For c_n and d_n, fractions with the same respective denominators as those of a_n and b_n are found, with $m[\psi_n'(x)\zeta_n(x)-\psi_n(x)\zeta_n'(x)]$ as a common numerator. At this point we have completed the solution for the scattering of electromagnetic waves by a sphere whose radius is a and whose index of refraction is m. The electric and magnetic field vectors specified in Eqs. (3.15) and (3.16) at any point inside or outside the sphere are now expressed in terms of known mathematical functions given by Eqs. (4.16).(4.18). Kattawar and Plass [10] have presented a_n and b_n of Eq. (4.22) in forms which are more efficient for computational purposes.

We have assumed up to this point for simplicity that the suspending medium is a vacuum. Now let the outside medium and the sphere have refractive indices m_2 (real) and m_1 (maybe complex), respectively. Replacing the m by m_1/m_2 and the wavenumber k by m_2k(vacuum), the results in Eq. (4.22) can be generalized to cases where a sphere is suspended in a medium.

5. THE FAR-FIELD SOLUTION

We shall now consider the scattered field at very large distances from the sphere. For light scattering, all observations are normally carried out in the far-field zone. In the far field, the Hankel functions denoted in Eq. (3.41) reduce to

$$\zeta_n(kr) \approx i^{n+1} e^{-ikr}, \qquad kr \gg 1. \tag{5.1}$$

Thus, Eq. (4.18) yields the following forms

$$ru^s \approx -\frac{ie^{-ikr}\cos\phi}{k}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}a_n P_n^1(\cos\theta), \tag{5.2a}$$

$$rv^s \approx -\frac{ie^{-ikr}\sin\phi}{k}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}b_n P_n^1(\cos\theta). \tag{5.2b}$$

The three components of the electric and magnetic field vectors in Eqs. (3.15) and (3.16) are given by

$$E_r^s = H_r^s \approx 0, \tag{5.3a}$$

$$E_\theta^s = H_\phi^s \approx \frac{-i}{kr}e^{-ikr}\cos\phi\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left[a_n\frac{dP_n^1(\cos\theta)}{d\theta} + b_n\frac{P_n^1(\cos\theta)}{\sin\theta}\right], \tag{5.3b}$$

$$-E_\phi^s = H_\theta^s \approx \frac{-i}{kr}e^{-ikr}\sin\phi\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left[a_n\frac{P_n^1(\cos\theta)}{\sin\theta} + b_n\frac{dP_n^1(\cos\theta)}{d\theta}\right]. \tag{5.3c}$$

It is clear that the radial components E_r^s and H_r^s may be neglected in the far-field zone. To simplify Eq. (5.3), we define the two scattering functions in the following forms:

$$S_1(\theta)=\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left[a_n\pi_n(\cos\theta)+b_n\tau_n(\cos\theta)\right], \tag{5.4a}$$

$$S_2(\theta)=\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left[b_n\pi_n(\cos\theta)+a_n\tau_n(\cos\theta)\right], \tag{5.4b}$$

where

$$\pi_n(\cos\theta)=\frac{1}{\sin\theta}P_n^1(\cos\theta), \tag{5.5a}$$

$$\tau_n(\cos\theta)=\frac{d}{d\theta}P_n^1(\cos\theta). \tag{5.5b}$$

Thus,

$$E_\theta^s=-\frac{i}{kr}e^{-ikr}\cos\phi\, S_2(\theta), \tag{5.6a}$$

$$-E_\phi^s=-\frac{i}{kr}e^{-ikr}\sin\phi\, S_1(\theta). \tag{5.6b}$$

These fields represent an outgoing spherical wave with amplitude and state of polarization as functions of the scattering angle θ. It is convenient to define the perpendicular and parallel components of the electric field as E_r and E_l, respectively. As seen in Fig. 2, the scattered perpendicular and parallel electric fields are given by

$$E_r^s=-E_\phi^s, \tag{5.7a}$$

$$E_l^s=E_\theta^s, \tag{5.7b}$$

and the normalized incident electric vector [see Eq. (4.1)] may be decomposed into

$$E_r^i=e^{-ikz}\sin\phi, \tag{5.8a}$$

$$E_l^i=e^{-ikz}\cos\phi. \tag{5.8b}$$

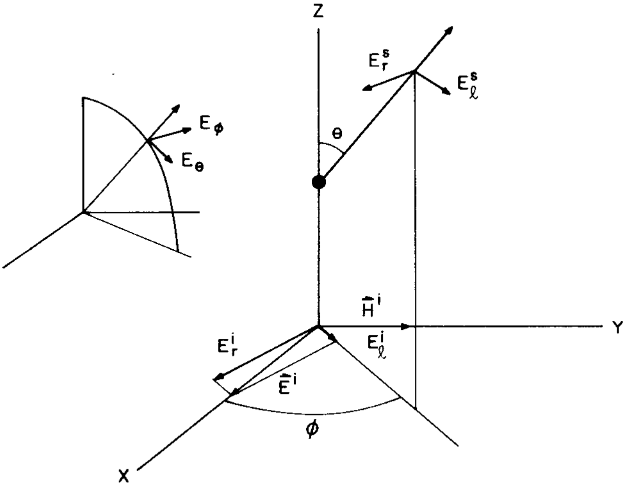

FIG. 2. Decomposition of the incident and scattered electric vectors into perpendicular and parallel components.

Equation (5.6) can then be expressed as

$$\begin{pmatrix} E_l^s \\ E_r^s \end{pmatrix} = \frac{e^{-ikr+ikz}}{ikr} \begin{pmatrix} S_2(\theta) & 0 \\ 0 & S_1(\theta) \end{pmatrix} \begin{pmatrix} E_l^i \\ E_r^i \end{pmatrix}. \tag{5.9}$$

Although van de Hulst [6, pp. 124–126] has implicitly shown the above relationship, it is here proven rigorously for the first time. It is the fundamental equation for the study of scattered radiation by spheres including polarization.

The scattered intensity components I^s in the far-field zone can now be written in terms of the incident intensity components I^i in the form

$$I_l^s = I_l^i \frac{i_2}{k^2 r^2}, \tag{5.10a}$$

$$I_r^s = I_r^i \frac{i_1}{k^2 r^2}, \tag{5.10b}$$

where

$$i_1(\theta) = |S_1(\theta)|^2, \tag{5.11a}$$

$$i_2(\theta) = |S_2(\theta)|^2 \tag{5.11b}$$

are called the intensity functions for the perpendicular and parallel components, respectively. Each of these components of the scattered light can be thought of as arising from the component of the incident beam polarized in the same direction. The computational problem involved in the Mie problem is to compute i_1 and i_2 as functions of the scattering angle, the index of refraction m and the particle size parameter $x = 2\pi a/\lambda$.

If natural unpolarized light of intensity I_0 is incident on a sphere, the intensity of scattered light in any direction may be obtained in a similar way to that in Eq. (5.10). It is partially linearly polarized, and its intensity is

$$I = I_0 \frac{i_1 + i_2}{2k^2 r^2}. \tag{5.12}$$

The degree of linear polarization is then

$$P = \frac{i_1 - i_2}{i_1 + i_2}. \tag{5.13}$$

6. EXTINCTION PARAMETERS

In the far-field zone, we would like to evaluate the reduction of the incident energy due to the absorption and scattering of light by a sphere. We consider incident light polarized linearly in the perpendicular direction. The scattered electric field is given by

$$E_r^s = \frac{e^{-ikr+ikz}}{ikr} S_1(\theta) E_r^i. \tag{6.1}$$

Next we consider a point (x, y, z) in the forward direction, i.e., $\theta \approx 0$. In the far field, since x, $y \ll z$, we have in the forward direction

$$r = \left(x^2 + y^2 + z^2\right)^{1/2} \approx z + \frac{x^2 + y^2}{2z}. \tag{6.2}$$

Upon superimposing the incident and scattered electric fields in the forward direction, we obtain

$$E_r^i + E_r^s \approx E_r^i \left\{ 1 + \frac{S_1(0)}{ikz} e^{-ik(x^2+y^2)/2z} \right\}. \tag{6.3}$$

The far-field combined flux density in the forward direction is then proportional to (note that $z \gg x, y$)

$$|E_r^i + E_r^s|^2 \approx |E_r^i|^2 \left\{ 1 + \frac{2}{kz} \operatorname{Re}\left[\frac{S_1(0)}{i} e^{-ik(x^2+y^2)/2z} \right] \right\}, \tag{6.4}$$

where Re[ ] represents the real part of the argument.

Integrating the combined flux density over the cross-sectional area of a sphere whose radius is a, we obtain the total power of the combined image:

$$\frac{1}{|E_r^i|^2} \int\int |E_r^i + E_r^s|^2 \, dx\, dy = \pi a^2 + \sigma_{\text{ext}}, \tag{6.5}$$

where the first term on the right-hand side represents the cross-sectional area of the sphere. The physical interpretation of the second term σ_{ext} is that the total light received in the forward direction is reduced by the presence of the sphere and the amount of the reduction is as if an area σ_{ext} of the objective had been covered up. The double integral over $dx\, dy$ by which σ_{ext} is defined contains two Fresnel integrals, and if the limits are assumed to extend to ∞ we get

$$\int\int_{-\infty}^{+\infty} e^{-ik(x^2+y^2)/2z} \, dx\, dy = \frac{2\pi z}{ik}. \tag{6.6}$$

Thus, the extinction cross-section is

$$\sigma_{\text{ext}} = \frac{4\pi}{k^2} \operatorname{Re}[S(0)]. \tag{6.7}$$

We note here that in the forward direction

$$S_1(0) = S_2(0) = S(0) = \frac{1}{2} \sum_{n=1}^{\infty} (2n+1)(a_n + b_n). \tag{6.8}$$

The fact that there is only one $S(0)$ is owing to the symmetry of the forward scattering, which implies that the extinction is independent of the state of polarization of the incident light. Note that Eq. (6.8) is valid only when a sphere is isotropic and homogeneous. Discussions of the use of the forward scattering to recover the particle size have been given by Fymat [11].

Furthermore, we define the extinction efficiency for a sphere with a radius a as

$$Q_{\mathrm{ext}}=\frac{\sigma_{\mathrm{ext}}}{\pi a^2}=\frac{2}{x^2}\sum_{n=1}^{\infty}(2n+1)\,\mathrm{Re}(a_n+b_n), \tag{6.9}$$

where $x=ka$ (as defined earlier) is called the size parameter. Chýlek [12] derived the asymptotic limit ($x\to\infty$) of Q_{ext}, which is equal to 2 from the above equation.

The scattering cross-section can be derived by the following procedures. From Eq. (5.6), the flux density of the scattered light in an arbitrary direction is given by

$$F(\theta,\phi)=\frac{F_0}{k^2r^2}\left[i_2(\theta)\cos^2\phi+i_1(\theta)\sin^2\phi\right] \tag{6.10}$$

with $F_0=1$ (unit incident amplitude). The total power of the scattered light is therefore

$$W=\int_0^{2\pi}\int_0^{\pi}F(\theta,\phi)r^2\sin\theta\,d\theta\,d\phi, \tag{6.11}$$

where $\sin\theta\,d\theta\,d\phi$ is the differential solid angle $d\omega$, and $r^2\,d\omega$ denotes the differential area. Hence the scattering cross-section may be defined as

$$\sigma_{\mathrm{sca}}=\frac{W}{F_0}=\frac{\pi}{k^2}\int_0^{\pi}\left[i_1(\theta)+i_2(\theta)\right]\sin\theta\,d\theta. \tag{6.12}$$

As in the extinction case, we define the scattering efficiency for a sphere

$$Q_{\mathrm{sca}}=\frac{\sigma_{\mathrm{sca}}}{\pi a^2}=\frac{1}{x^2}\int_0^{\pi}\left[i_1(\theta)+i_2(\theta)\right]\sin\theta\,d\theta. \tag{6.13}$$

We note that

$$\int_0^{\pi}\left(\frac{dP_n^1}{d\theta}\frac{dP_m^1}{d\theta}+\frac{1}{\sin^2\theta}P_n^1P_m^1\right)\sin\theta\,d\theta=\begin{cases}0 & \text{if } n\neq m,\\ \dfrac{2n(n+1)}{2n+1}\dfrac{(n+1)!}{(n-1)!} & \text{if } n=m,\end{cases} \tag{6.14}$$

and that

$$\int_0^{\pi}\left(\frac{P_n^1}{\sin\theta}\frac{dP_m^1}{d\theta}+\frac{P_m^1}{\sin\theta}\frac{dP_n^1}{d\theta}\right)\sin\theta\,d\theta=0. \tag{6.15}$$

The scattering efficiency can be evaluated with the help of these two equations to yield

$$Q_{\text{sca}}=\frac{2}{x^2}\sum_{n=1}^{\infty}(2n+1)\left(|a_n|^2+|b_n|^2\right). \tag{6.16}$$

Finally, the absorption cross-section and efficiency of a sphere can be calculated from

$$\sigma_{\text{abs}}=\sigma_{\text{ext}}-\sigma_{\text{sca}},$$

$$Q_{\text{abs}}=Q_{\text{ext}}-Q_{\text{sca}}. \tag{6.17}$$

For an absorbing sphere, it is convenient to define the index of refraction as $m=m_r-im_i$, with m_r and m_i representing the real and imaginary parts of the refractive index.

7. STOKES PARAMETERS AND PHASE MATRIX

Electromagnetic waves scattered by spheres are in general elliptically polarized. The electric vector is transverse [see Eq. (5.9)] and may be represented by

$$\mathbf{E}=\text{Re}(E_l\mathbf{l}+E_r\mathbf{r}), \tag{7.1}$$

where $\mathbf{l}$ and $\mathbf{r}$ are unit vectors and E_l and E_r are complex, oscillating functions from which the Stokes parameters are defined by

$$I=E_lE_l^*+E_rE_r^*, \tag{7.2a}$$

$$Q=E_lE_l^*-E_rE_r^*, \tag{7.2b}$$

$$U=E_lE_r^*+E_rE_l^*, \tag{7.2c}$$

$$V=i(E_lE_r^*-E_rE_l^*). \tag{7.2d}$$

The asterisk denotes the complex conjugate. These quantities are real numbers and describe completely the state of polarization of electromagnetic waves. Radiative-transfer theory involving polarization normally employs Stokes parameters in the formulation. Recently, however, Fymat and Vasudevan [13] showed that the radiative transfer of polarized light may be treated by utilizing the electric field vectors with the so-called Jones matrix, which preserves information on the phases of electromagnetic waves.

We may now define the Stokes parameters for the incident and scattered waves described by Eq. (5.9). We have

$$\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \mathbf{M} \begin{bmatrix} I_0 \\ Q_0 \\ U_0 \\ V_0 \end{bmatrix}, \tag{7.3}$$

where

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} & 0 & 0 \\ M_{12} & M_{11} & 0 & 0 \\ 0 & 0 & M_{33} & -M_{34} \\ 0 & 0 & M_{34} & M_{33} \end{bmatrix}, \tag{7.4}$$

$$M_{11} = \frac{1}{2k^2r^2}\left[S_1(\theta)S_1^*(\theta) + S_2(\theta)S_2^*(\theta)\right], \tag{7.5a}$$

$$M_{12} = \frac{1}{2k^2r^2}\left[S_2(\theta)S_2^*(\theta) - S_1(\theta)S_1^*(\theta)\right], \tag{7.5b}$$

$$M_{33} = \frac{1}{2k^2r^2}\left[S_2(\theta)S_1^*(\theta) + S_1(\theta)S_2^*(\theta)\right], \tag{7.5c}$$

$$-M_{34} = \frac{i}{2k^2r^2}\left[S_1(\theta)S_2^*(\theta) - S_2(\theta)S_1^*(\theta)\right]. \tag{7.5d}$$

Here **M** is called the transformation matrix of a single sphere. For incident unpolarized light ($Q_0 = U_0 = V_0 = 0$), Eq. (7.3) reduces to Eq. (5.12).

In conjunction with the transformation matrix, we can define a parameter **P**, called the phase matrix, in such a way that

$$\mathbf{M}(\theta) = C\mathbf{P}(\theta), \tag{7.6}$$

and that

$$\int_0^{2\pi}\int_0^{\pi}\frac{P_{11}(\theta)}{4\pi}\sin\theta\,d\theta\,d\phi=1. \tag{7.7}$$

On the basis of Eqs. (7.6) and (7.7), it is evident that

$$\begin{aligned}C&=\frac{1}{2}\int_0^{\pi}M_{11}(\theta)\sin\theta\,d\theta\\&=\frac{1}{4k^2r^2}\int_0^{\pi}\left[i_1(\theta)+i_2(\theta)\right]\sin\theta\,d\theta.\end{aligned} \tag{7.8}$$

According to the definition of the scattering cross-section in Eq. (6.12), the constant of proportionality

$$C=\frac{\sigma_{\text{sca}}}{4\pi r^2}. \tag{7.9}$$

Thus,

$$\frac{P_{11}}{4\pi}=\frac{1}{2k^2\sigma_{\text{sca}}}(i_1+i_2), \tag{7.10a}$$

$$\frac{P_{12}}{4\pi}=\frac{1}{2k^2\sigma_{\text{sca}}}(i_2-i_1), \tag{7.10b}$$

$$\frac{P_{33}}{4\pi}=\frac{1}{2k^2\sigma_{\text{sca}}}(i_3+i_4), \tag{7.10c}$$

$$-\frac{P_{34}}{4\pi}=\frac{i}{2k^2\sigma_{\text{sca}}}(i_4-i_3), \tag{7.10d}$$

where

$$i_1=S_1S_1^*=|S_1|^2, \tag{7.11a}$$

$$i_2=S_2S_2^*=|S_2|^2, \tag{7.11b}$$

$$i_3=S_2S_1^*, \tag{7.11c}$$

$$i_4=S_1S_2^*. \tag{7.11d}$$

The scattering phase matrix for a single homogeneous sphere is then

$$\mathbf{P}=\begin{bmatrix} P_{11} & P_{12} & 0 & 0 \\ P_{12} & P_{11} & 0 & 0 \\ 0 & 0 & P_{33} & -P_{34} \\ 0 & 0 & P_{34} & P_{33} \end{bmatrix}. \tag{7.12}$$

In general, if no assumption is made on the shape and position of the scatterer, the scattering phase matrix consists of 16 independent elements. For a single sphere, it is clear that the independent elements reduce to only four. Analytic relations between the elements of the scattering phase matrix have been derived by Abhyankar and Fymat [14].

8. MIE SCATTERING FOR A SAMPLE OF SPHERICAL PARTICLES

All the developments discussed in the previous sections are concerned with the scattering of electromagnetic waves by a single homogeneous sphere. We shall now extend these developments to a sample of cloud or aerosol particles so that practical equations for the calculations of extinction parameters and phase functions may be derived. We *assume* that particles are sufficiently far from each other and that the distance between them is much greater than the incident wavelengths. Thus, it is possible to study the scattering by one particle without reference to the other ones. Consequently, intensities scattered by various particles may be added without regard to the phase of the scattered waves. This particular kind of scattering is called *independent scattering*, and it is assumed in all the following discussions.

We consider a sample of cloud particles whose size spectrum can be described by $dn(a)/da$ (in units, say, of $\text{cm}^{-3}\mu\text{m}^{-1}$). Assume that size range of particles is from a_1 to a_2; then the total number of particles is given by

$$N=\int_{a_1}^{a_2}\frac{dn(a)}{da}\,da. \tag{8.1}$$

With the particle size distribution prescribed, we can define the extinction and scattering parameters for a sample of particles. The extinction and scattering coefficients (with dimensions of length^{-1}) are defined respectively as follows:

$$\beta_{\text{ext}}=\int_{a_1}^{a_2}\sigma_{\text{ext}}(a)\frac{dn(a)}{da}\,da, \tag{8.2}$$

$$\beta_{\mathrm{sca}}=\int_{a_1}^{a_2}\sigma_{\mathrm{sca}}(a)\frac{dn(a)}{da}\,da. \tag{8.3}$$

The single-scattering albedo for a sample of particles is now given by

$$\tilde{\omega}_0=\frac{\beta_{\mathrm{sca}}}{\beta_{\mathrm{ext}}}. \tag{8.4}$$

We now define the phase matrix for a sample of particles. Since the phase matrix is a physical parameter describing the scattered intensity and polarization state for a sample of particles, it is independent of the particle size distribution. Hence, we rearrange Eq. (7.10a) and perform particle-size integration to obtain

$$\frac{P_{11}}{4\pi}\int_{a_1}^{a_2}\sigma_{\mathrm{sca}}\frac{dn(a)}{da}\,da=\frac{1}{2k^2}\int_{a_1}^{a_2}\left[i_1(a)+i_2(a)\right]\frac{dn(a)}{da}\,da, \tag{8.5}$$

i.e.,

$$\frac{P_{11}}{4\pi}=\frac{1}{2k^2\beta_{\mathrm{sca}}}\int_{a_1}^{a_2}\left[i_1(a)+i_2(a)\right]\frac{dn(a)}{da}\,da. \tag{8.6}$$

Similarly, we have

$$\frac{P_{12}}{4\pi}=\frac{1}{2k^2\beta_{\mathrm{sca}}}\int_{a_1}^{a_2}\left[i_2(a)-i_1(a)\right]\frac{dn(a)}{da}\,da, \tag{8.7}$$

$$\frac{P_{33}}{4\pi}=\frac{1}{2k^2\beta_{\mathrm{sca}}}\int_{a_1}^{a_2}\left[i_3(a)+i_4(a)\right]\frac{dn(a)}{da}\,da, \tag{8.8}$$

$$-\frac{P_{34}}{4\pi}=\frac{i}{2k^2\beta_{\mathrm{sca}}}\int_{a_1}^{a_2}\left[i_4(a)-i_3(a)\right]\frac{dn(a)}{da}\,da. \tag{8.9}$$

Note here that i_j $(j=1,2,3,4)$ are functions of the particle radius a, the index of refraction m, the incident wavelength λ and the scattering angle θ. At this point, the scattering of electromagnetic waves by large homogeneous spheres, the so called Mie scattering, is completely described.

9. SUMMARY

In this paper, I have developed comprehensively and coherently the theory of light scattering by homogeneous spheres. The theory begins with the Maxwell equations, from which the vector wave equations for electric and magnetic fields in a periodic field are derived. Solutions of the vector wave equations are obtained by defining two potentials satisfied by the scalar wave equation. The solution of the scalar wave equation in spherical coordinates is developed in terms of the known mathematical functions.

To determine the scattered fields of electromagnetic waves, the incident fields are expressed in spherical coordinates, in which expansions of known mathematical functions can be carried out. The two potentials of the incident field are subsequently determined from the decomposition of the scalar wave equation. The boundary conditions are now required for the determination of the unknown coefficients in the two potentials of the internal and scattered fields.

Far-field solutions for the electric field are given from the two potentials of the scattered field. It is shown that scattered electric field is transverse, the radial component being negligibly small. The scattered electric field is transformed to spherical coordinates, so that the scattered electric field is represented by components perpendicular and parallel to the plane containing the incident and scattered waves. At this point, the scattered electric-field components can be expressed in terms of the incident field components and the intensity functions. It follows that the extinction and scattering parameters in the far field are defined. Extinction and scattering efficiencies are given by the scattering coefficients appearing in the two potentials. A unique definition of the phase matrix for a homogeneous sphere is derived.

Finally, the scattering theory is applied to a sample of homogeneous spheres of various sizes to give the formulas for the extinction and scattering coefficients and phase matrix elements. The development here is subject to the independent-scattering assumption.

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